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Exam on Quantum Field Theory I

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Problem 1 (11 points)

The action for a complex scalar field $\phi(x)$ is given by

$$S = \int d^4x \left(\partial_\mu \phi^\dagger(x) \partial^\mu \phi(x) - M^2 \phi^\dagger(x) \phi(x) \right). \qquad (1)$$

The mode expansion of $\phi(x)$ is given by

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a(p) e^{-ipx} + b^\dagger(p) e^{ipx} \right) \Big|_{p^0 = E_p},$$

where the creation and annihilation operators satisfy the commutation relations

$$[a(p), a^\dagger(q)] = [b(p), b^\dagger(q)] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}).$$

a) For a theory of several fields χ_k the canonical stress-energy tensor is

$$T^{\mu\nu} = \sum_k \frac{\partial \mathcal{L}}{\partial(\partial_\mu \chi_k)} \partial^\nu \chi_k - \eta^{\mu\nu} \mathcal{L}.$$

Show that $\partial_\mu T^{\mu\nu} = 0$ on shell for the theory of a complex scalar field given by (1).

$$T^{\mu\nu} = \partial^\mu \phi^\dagger \partial^\nu \phi + \partial^\mu \phi \partial^\nu \phi^\dagger - \eta^{\mu\nu} \partial_\alpha \phi^\dagger \partial^\alpha \phi + \eta^{\mu\nu} M^2 \phi^\dagger \phi$$

$$\partial_\mu T^{\mu\nu} = \partial_\mu(\partial^\mu \phi^\dagger \partial^\nu \phi) + \partial_\mu(\partial^\mu \phi \partial^\nu \phi^\dagger) - \partial^\nu(\partial_\alpha \phi^\dagger \partial^\alpha \phi) + \partial^\nu(M^2 \phi^\dagger \phi)$$

$$= \partial^2 \phi^\dagger \partial^\nu \phi + \underline{\partial^\mu \phi^\dagger \partial_\mu \partial^\nu \phi} + \partial^2 \phi \partial^\nu \phi^\dagger + \underline{\underline{\partial^\mu \phi \partial_\mu \partial^\nu \phi^\dagger}}$$

$$- \underline{\underline{\partial^\nu \partial_\alpha \phi^\dagger \partial^\alpha \phi}} - \underline{\partial_\alpha \phi^\dagger \partial^\nu \partial^\alpha \phi} + M^2 \phi^\dagger \partial^\nu \phi + M^2 \phi \partial^\nu \phi^\dagger$$

$$= \underbrace{(\partial^2 \phi^\dagger + M^2 \phi^\dagger)}_{=0,\ \text{e.o.m.}} \partial^\nu \phi + \underbrace{(\partial^2 \phi + M^2 \phi)}_{=0,\ \text{e.o.m.}} \partial^\nu \phi^\dagger = 0$$

We see that on-shell, i.e. when imposing the Klein-Gordon eq. - the equation of motion for the free scalar field - $(\partial^2 + m^2)\phi = 0$ to be true, the stress-energy tensors total derivative vanishes.

b) From the stress-tensor components we define conserved charges as $Q^\nu = \int d^3x \, T^{0\nu}$. Show that the charges Q^i for $i \in \{1,2,3\}$ are given by

$$Q^i = \int \frac{d^3p}{(2\pi)^3} p^i \left(a^\dagger(p) a(p) + b^\dagger(p) b(p) \right).$$

Hint: Perform the integral $\int d^3x$ before performing any momentum integral.

From part a), we can deduce T^{0i} with $i \in \{1,2,3\}$ to be

$$T^{0i} = \partial^0 \phi^\dagger \partial^i \phi + \partial^0 \phi \, \partial^i \phi^\dagger = \partial_t \phi^\dagger \frac{\partial}{\partial x_i} \phi + \partial_t \phi \frac{\partial}{\partial x_i} \phi^\dagger, \text{ where}$$

$$\partial_t \phi = i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{E_p}{2}} \left(b^\dagger(p) e^{ipx} - a(p) e^{-ipx} \right)$$

$$\partial^i \phi = -i \int \frac{d^3p}{(2\pi)^3} \frac{p^i}{\sqrt{2E_p}} \left(a(p) e^{-ipx} - b^\dagger(p) e^{ipx} \right)$$

Inserting the above expressions into the charge, we get

$$Q^i = \int d^3x \left(\partial_t \phi^\dagger \partial^i \phi + \partial_t \phi \, \partial^i \phi^\dagger \right) = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} p^i \Big(a^\dagger(p) a(p) - a^\dagger(p) b^\dagger(-p)$$

$$+ b(p) a(-p) + b(p) b^\dagger(p) + b^\dagger(p) b(p) - b^\dagger(p) a^\dagger(-p) - a(p) b(-p) + a(p) a^\dagger(p) \Big)$$

$$= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} p^i \left(2a^\dagger(p) a(p) + 2 b^\dagger(p) b(p) + 2 (2\pi)^3 \delta^{(3)}(0) \right)$$

$$- \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \Big(\underbrace{p^i \left(a^\dagger(p) b^\dagger(-p) + a^\dagger(-p) b^\dagger(p) \right)}_{\text{antisymmetric under } p^i \to -p^i} + \underbrace{p^i \left(a(p) b(-p) + a(-p) b(p) \right)}_{\text{antisymmetric under } p^i \to -p^i} \Big)$$

$$= \int \frac{d^3p}{(2\pi)^3} p^i \left(a^\dagger(p) a(p) + b^\dagger(p) b(p) \right) - \int d^3p \, \delta^{(3)}(0)$$

We could get an overall minus sign in front of the charge, which would be no issue since we allow charges to take positive and negative values and which we call which is up to us. The last term is clearly divergent (and constant). It needs to be dropped before we can let the charge operator act on physical states.

c) Consider the action of Q^i on the states $a^\dagger(k)|0\rangle$ and $b^\dagger(q)|0\rangle$ and hence give a physical interpretation for the charges Q^i.

$$Q^i a^\dagger(k)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\, p^i \left(a^\dagger(p)\, a(p) + b^\dagger(p)\, b(p)\right) a^\dagger(k)|0\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3}\, p^i \left(a^\dagger(p)\left(a^\dagger(k)\, a(p) + (2\pi)^3 \delta^{(3)}(p-k)\right) + a^\dagger(k)\, b^\dagger(p)\, b(p)\right)|0\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3}\, p^i\, a^\dagger(p)\, (2\pi)^3 \delta^{(3)}(p-k)\, |0\rangle = k^i\, a^\dagger(k)\, |0\rangle$$

$$Q^i b^\dagger(q)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\, p^i \left(a^\dagger(p)\, a(p) + b^\dagger(p)\, b(p)\right) b^\dagger(q)|0\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3}\, p^i \left(b^\dagger(q)\, a^\dagger(p)\, a(p) + b^\dagger(p)\left(b^\dagger(q)\, b(q) + (2\pi)^3 \delta^{(3)}(p-q)\right)\right)|0\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3}\, p^i\, b^\dagger(p)\, (2\pi)^3 \delta^{(3)}(p-q)\, |0\rangle = q^i\, b^\dagger(q)\, |0\rangle,$$

where we used that the annihilation operators $a(p), b(p)$ destroy the vacuum

$$a(p)|0\rangle = b(p)|0\rangle = 0 \qquad \forall p$$

We find that Q^i's eigenvalues corresponding to the 1-particle states $a^\dagger(k)|0\rangle$ and $b^\dagger(q)|0\rangle$ are the spatial components of the momenta k and q.

Problem 2 (6 points)

The Lagrangian density for a massless vector field A^μ is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \qquad \text{where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \qquad (2)$$

a) What problem does arise if one tries to quantise the theory given by eq. (2) by enforcing canonical equal-time commutation relations

$$[A^\mu(t,\vec{x}), \Pi_\nu(t,\vec{y})] = i\, \delta^\mu{}_\nu\, \delta^{(3)}(\vec{x}-\vec{y})\ ?$$

The Lagrangian in eq. (2) is not suitable for quantisation because it produces an unphysical conjugate momentum density. Specifically, Π_μ turns out to be

$$\Pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}^\mu} = F_{\mu 0},$$

which yields immediately that $\Pi_0 = F_{00} = 0$ since $F_{\mu\nu}$ is antisymmetric.
Hence we must concede that (A^μ, Π_μ) are no good canonical variables.

b) Sketch briefly how the problem can be overcome.

The above problem only cropped up because of the particular gauge we were working in. We can make use of the redundancy in the physical description of gauge fields such as A^μ by imposing a gauge fix such as Lorenz gauge which alters the Lagrangian and therefore the canonically conjugate momentum density without changing the equations of motions, i.e. the physical behaviour of the fields.

Problem 3 (12 points)

The Lagrangian density for a free Dirac fermion is given by

$$\mathcal{L}(x) = \bar{\Psi}(x)\left(i\gamma^\mu \partial_\mu - m\right)\Psi(x). \tag{3}$$

Consider a Lorentz transformation where we keep the coordinate axes fixed and only transform the fields, i.e.

$$x^\mu \to x^\mu \text{ (unchanged)}, \quad \partial^\mu \to \partial^\mu \text{ (unchanged)}, \quad \Psi(x) \to \Lambda_{\frac{1}{2}} \Psi(\Lambda^{-1}x),$$

where $\Lambda_{\frac{1}{2}} = \exp\left(\frac{i}{2}\omega_{\mu\nu} S^{\mu\nu}\right)$ with $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$. We also work in a basis where

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

for $i \in \{1,2,3\}$, where $\mathbb{1}_2$ is the 2x2 unit matrix and σ^i are the Pauli matrices.

a) Show that $(S^{\mu\nu})^\dagger \gamma^0 = \gamma^0 S^{\mu\nu}$.

$$(S^{\mu\nu})^\dagger \gamma^0 = \frac{i}{4}\left[(\gamma^\nu)^\dagger, (\gamma^\mu)^\dagger\right]\gamma^0 = \frac{i}{4}\left((\gamma^\nu)^\dagger(\gamma^\mu)^\dagger\gamma^0 - (\gamma^\mu)^\dagger(\gamma^\nu)^\dagger\gamma^0\right)$$

$$= \frac{i}{4}\left(\underbrace{\gamma^0\gamma^0}_{\mathbb{1}_4}(\gamma^\nu)^\dagger\gamma^0\gamma^0(\gamma^\mu)^\dagger\gamma^0 - \gamma^0\gamma^0(\gamma^\mu)^\dagger\gamma^0\gamma^0\underbrace{(\gamma^\nu)^\dagger\gamma^0}_{\gamma^0\gamma^\nu}\right) = \gamma^0\frac{i}{4}[\gamma^\mu,\gamma^\nu] = \gamma^0 S^{\mu\nu},$$

where we used that $(\gamma^0)^2 = \mathbb{1}_4$, which follows directly from the defining relation of the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}_4$ for $\mu = \nu = 0$ and that $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$ which holds specifically for our choice of gamma matrices, where $(\gamma^0)^\dagger = \gamma^0$ and $(\gamma^i)^\dagger = -\gamma^i$. Together with the Clifford algebra's anticommutator, this yields

$$(\gamma^\mu)^\dagger = \begin{cases} \gamma^\mu = \gamma^0\gamma^0\gamma^\mu = \gamma^0\gamma^\mu\gamma^0 & \text{for } \mu = 0 \\ -\gamma^\mu = -\gamma^0\gamma^0\gamma^\mu = \gamma^0\gamma^\mu\gamma^0 & \text{for } \mu \in \{1,2,3\} \end{cases}$$

b) Using the result from part a) and the fact that $(\Lambda_{\frac{1}{2}})^{-1}\gamma^\mu\Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu\gamma^\nu$ show that under a Lorentz transformation the Lagrangian density transforms as $\mathcal{L}(x) \longrightarrow \mathcal{L}(\Lambda^{-1}x)$.

$$\mathcal{L}(x) \xrightarrow{\Lambda_{\frac{1}{2}}} (\Lambda_{\frac{1}{2}}\Psi(\Lambda^{-1}x))^\dagger\gamma^0(i\gamma^\mu\partial_\mu - m)\Lambda_{\frac{1}{2}}\Psi(\Lambda^{-1}x)$$

$$= \Psi^\dagger(\Lambda^{-1}x)\, e^{-\frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})^\dagger}\gamma^0(i\gamma^\mu\partial_\mu - m)\Lambda_{\frac{1}{2}}\Psi(\Lambda^{-1}x)$$

$$= \bar{\Psi}(\Lambda^{-1}x)\underbrace{e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}}_{(\Lambda_{\frac{1}{2}})^{-1}}(i\gamma^\mu\partial_\mu - m)\Lambda_{\frac{1}{2}}\Psi(\Lambda^{-1}x)$$

$$= \bar{\Psi}(\Lambda^{-1}x)(i\Lambda^\mu{}_\nu\gamma^\nu\partial_\mu - m)\Psi(\Lambda^{-1}x) = \mathcal{L}(\Lambda^{-1}x),$$

where we used that $e^{(S^{\mu\nu})^\dagger}\gamma^0 = \sum_{n=0}^{\infty}\frac{((S^{\mu\nu})^\dagger)^n}{n!}\gamma^0 \overset{a)}{=} \gamma^0\sum_{n=0}^{\infty}\frac{(S^{\mu\nu})^n}{n!} = \gamma^0 e^{S^{\mu\nu}}$ and that $\Lambda^\mu{}_\nu = (\Lambda^{-1})^\nu{}_\mu$ in the last step.

c) Define $\Psi_\pm = P_\pm\Psi$ where $P_\pm = \frac{1}{2}(\mathbb{1}_4 \pm \gamma^5)$ and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. Show that $\overline{\Psi_\pm} = \bar{\Psi}P_\mp$. Rewrite the Lagrangian density (3) in terms of Ψ_+ and Ψ_-. Derive the equation of motion for Ψ_+ and Ψ_-.

Note: You can use identities involving γ^5 without proof.

$$(\gamma^5)^\dagger = -i(\gamma^3)^\dagger(\gamma^2)^\dagger(\gamma^1)^\dagger(\gamma^0)^\dagger = -i(-\gamma^3)(-\gamma^2)(-\gamma^1)\gamma^0 = i\gamma^3\gamma^2\gamma^1\gamma^0 = (-1)^3 i\gamma^0\gamma^3\gamma^2\gamma^1$$

$$= (-1)^5 i\gamma^0\gamma^1\gamma^3\gamma^2 = (-1)^6 i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5$$

$$\overline{\Psi_\pm} = (\Psi_\pm)^\dagger \gamma^0 = (P_\pm \Psi)^\dagger \gamma^0 = \Psi^\dagger P_\pm^\dagger \gamma^0 = \Psi^\dagger \frac{1}{2}(\mathbb{1}_4 \pm (\gamma^5)^\dagger)\gamma^0 = \Psi^\dagger \gamma^0 \frac{1}{2}(\mathbb{1}_4 \pm (-1)^3 \gamma^5)$$

$$= \overline{\Psi} \frac{1}{2}(\mathbb{1}_4 \mp \gamma^5) = \overline{\Psi} P_\mp$$

To rewrite the Lagrangian in eq. (3) in terms of Ψ_+ and Ψ_-, we note that

$$P_+ + P_- = \frac{1}{2}(\mathbb{1}_4 + \gamma^5) + \frac{1}{2}(\mathbb{1}_4 - \gamma^5) = \mathbb{1}_4 \implies \Psi_+ + \Psi_- = P_+\Psi + P_-\Psi = (P_+ + P_-)\Psi = \Psi$$

$$\overline{\Psi} = \Psi^\dagger \gamma^0 = (\Psi_+ + \Psi_-)^\dagger \gamma^0 = \Psi_+^\dagger \gamma^0 + \Psi_-^\dagger \gamma^0 = \overline{\Psi_+} + \overline{\Psi_-}$$

$$\mathcal{L} = \overline{\Psi_+}(i\gamma^\mu\partial_\mu - m)\Psi_+ + \overline{\Psi_+}(i\gamma^\mu\partial_\mu - m)\Psi_- + \overline{\Psi_-}(i\gamma^\mu\partial_\mu - m)\Psi_+ + \overline{\Psi_-}(i\gamma^\mu\partial_\mu - m)\Psi_-$$

$$= \overline{\Psi_+} i\gamma^\mu\partial_\mu \Psi_+ - \overline{\Psi_+} m \Psi_+ + \overline{\Psi_+} i\gamma^\mu\partial_\mu \Psi_- - \overline{\Psi_+} m \Psi_- + \overline{\Psi_-} i\gamma^\mu\partial_\mu \Psi_+ - \overline{\Psi_-} m \Psi_+ + \overline{\Psi_-} i\gamma^\mu\partial_\mu \Psi_- - \overline{\Psi_-} m \Psi_-$$

$$= \overline{\Psi_+} i\gamma^\mu\partial_\mu \Psi_+ + \overline{\Psi_-} i\gamma^\mu\partial_\mu \Psi_- - \overline{\Psi_+} m \Psi_- - \overline{\Psi_-} m \Psi_+,$$

where we used in the last step that $\overline{\Psi_\pm}\Psi_\pm = \overline{\Psi} P_\mp P_\pm \Psi = 0$ and $\overline{\Psi_\pm}\gamma^\mu\Psi_\mp = \overline{\Psi} P_\mp \gamma^\mu \Psi_\mp$ $= \overline{\Psi}\gamma^\mu P_\pm P_\mp \Psi = 0$ because $P_\pm P_\mp = \frac{1}{4}(\mathbb{1}_4 - (\gamma^5)^2) = \frac{1}{4}(\mathbb{1}_4 - \mathbb{1}_4) = 0$. Applying the Euler-Lagrange equation $\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi)} = \frac{\partial \mathcal{L}}{\partial \Psi}$ to the new Lagrangian gives

Ψ_+: $\partial_\mu \overline{\Psi_+} i\gamma^\mu + m\overline{\Psi_-} = 0$ $\qquad \overline{\Psi_+}$: $m\Psi_- = 0$

Ψ_-: $\partial_\mu \overline{\Psi_-} i\gamma^\mu + m\overline{\Psi_+} = 0$ $\qquad \overline{\Psi_-}$: $m\Psi_+ = 0$

d) The spinors Ψ_+ and Ψ_- correspond to two-component Weyl spinors. What are the properties of a particle which can be represented by a single Weyl spinor? For a particle to be described by a single Weyl spinor, the above equations of motion would have to be decoupled. We can see that this is only the case for $m = 0$.

Problem 4 (12 points)

Consider the following theory for a complex scalar field ϕ and a real scalar field σ

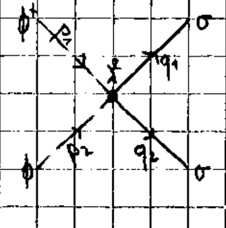
$$\mathcal{L} = \partial_\mu\phi^\dagger \partial^\mu \phi + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - M^2\phi^\dagger\phi - \frac{\mu^2}{2}\sigma^2 - \lambda\sigma\phi^\dagger\phi - \lambda^2\sigma^2\phi^\dagger\phi.$$

a) State the momentum space Feynman rules for calculating scattering amplitudes for this theory. (No calculation required.)

To compute the amplitude for a scattering process with n ingoing momenta $p_1, \dots, p_n$ and r outgoing momenta $q_1, \dots, q_r$ in this theory, we perform the following:

- Draw all relevant fully connected amputated diagrams with $(n+r)$ external points to given order in λ.
- Assign ingoing momenta labels p_k, $k \in \{1, \dots, n\}$ and outgoing momenta labels q_l, $l \in \{1, \dots, r\}$ and label momenta of internal lines with k_j.
- Each three-particle vertex gets a factor $-i\lambda$, each four-particle vertex $-i\lambda^2$.
- To each internal line, we associate a propagator
  $-\frac{i}{k_j^2 - m_\phi^2}$ if it is a ϕ field
  $-\frac{i}{k_j^2 - m_\sigma^2}$ if it is a σ field
- We then integrate over all undetermined internal momenta $\prod_j \int \frac{d^4 k_j}{(2\pi)^4}$.
- Sum up all diagrams divided by their symmetry factors and multiply by $\sqrt{Z}^{n+r}$ to given order in λ.

b) Draw all Feynman graphs to order λ^2 for the non-trivial scattering of $\phi^\dagger \phi \to \sigma\sigma$.

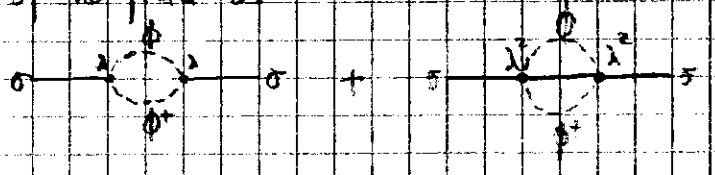


c) Now write down the matrix element iM for the process $\phi^\dagger \phi \to \sigma\sigma$ to order λ^2. What is the matrix element iM for the process $\phi\phi \to \sigma\sigma$ to order λ^2?

$iM = -i\lambda^2$

At order λ^2, the given Lagrangian does not allow for the process $\phi\phi \to \sigma\sigma$ to take place. Its scattering amplitude iM is zero.

c) Draw Feynman diagrams to and including order λ^4 which contribute to the 1PI self energy (i.e. the 1PI contributions to the propagator) of the field σ.



Problem 5 (9 points)

a) Discuss the difference between the bare mass m_0 and the physical mass m of an electron in Quantum Electrodynamics. Your answer should include

- which of both quantities is cutoff-dependent.
- their relation to the wavefunction renormalization Z_2 and the electron self-energy.

Also sketch how you would calculate Z_2 in perturbation theory.

Only the physical mass m is an observable; it is the rest mass of an electron as measured in experiments. By contrast m_0, the so called bare mass, is merely a parameters that specifies the Lagrangian. It can never be measured directly. Rather the Lagrangian produces measurable quantities, the scattering amplitudes, which depend on m_0.

This fact allows us to absorb the divergent quantity $\delta m := m - m_0$ completely into m_0, making the latter a function of the measured physical mass and the cutoff Λ, $m_0 = m_0(m, \Lambda)$.

To order α, the electron self-energy $\Sigma(\not{p})$ is given by $\Sigma(\not{p}) = m - m_0$ and the wavefunction renormalization Z_2 is defined as

$$Z_2 := \left(1 - \frac{d\Sigma(\not{p})}{d\not{p}}\right)^{-1}\Bigg|_{\not{p}=m}$$

b) Draw all Feynman diagrams in QED which contribute to vertex renormal. at order e_0^4 exactly.

This topic was addressed by our coverage of QFT. Part c) likewise.