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Mid-Semester Exam on Quantum Field Theory I

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Problem 1 (5 points)

Consider a free complex scalar field $\phi(x)$ with action

$$S = \int d^4x \left(\partial^\mu \phi^\dagger \partial_\mu \phi - m^2 \phi^\dagger \phi \right) \quad , \quad \phi = \phi(x) \text{ and } \phi^\dagger = \phi^\dagger(x). \qquad (1)$$

a) Derive the equation of motion for $\phi(x)$.

Equations of motions can be deduced from a system's action S by applying to it Hamilton's principle of the stationary action $\delta S = 0$. Hamilton's principle states that from all possible evolutions of a system, the (only) true evolution $\phi(t, \vec{x}) = \phi(x)$ is that which lies at a stationary point of the action functional $S[\phi(x)]$, i.e. a point where the variation $\delta S[\phi(x)]$ with respect to ϕ and $\partial_\mu \phi$ subject to $\delta\phi|_{\text{boundary}} = \delta \partial_\mu \phi|_{\text{boundary}} = 0$ is zero. This yields

$$0 \overset{!}{=} \delta S = \delta \left(\int d^4x \, \mathcal{L}(\phi, \partial_\mu \phi) \right) = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta \partial_\mu \phi \right)$$

$$= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu \delta \phi \right) = \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \Big|_{\text{boundary}}}_{0} + \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi$$

Since the variation of ϕ is in general unequal zero throughout spacetime, for the above to hold true requires

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0.$$

Inserting into this expression the Lagrangian in brackets in (1) and replacing ϕ by $\phi^\dagger$ since we are tasked to find the equation of motion for ϕ, results in

$$-m^2 \phi - \partial_\mu (\partial^\mu \phi) = 0 \implies (\partial^2 + m^2) \phi = 0$$

b) Derive the expression for the canonically conjugate momentum density to $\phi(x)$ and compute the Hamiltonian density $\mathcal{H}$.

The canonically conjugate momentum density is defined as

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi(x))} = \partial_t \phi^\dagger(x) = \dot{\phi}^\dagger(x)$$

The Hamiltonian density in turn is defined as

$$\mathcal{H} = \sum \pi_i(x) \dot{\phi}_i(x) - \mathcal{L},$$

where the sum runs over all independent pairs of conjugate fields and momentum densities. Since the complex scalar field has two degrees of freedom, $\phi(x)$ and $\phi^\dagger(x)$, this sum contributes with two terms

$$\mathcal{H} = \pi \dot{\phi} + \pi^\dagger \dot{\phi}^\dagger - \partial^\mu \phi^\dagger \partial_\mu \phi + m^2 \phi^\dagger \phi$$

$$= (\partial_t \phi^\dagger) \dot{\phi} + (\partial_t \phi) \dot{\phi}^\dagger - \partial_t \phi^\dagger \partial_t \phi + \vec{\nabla} \phi^\dagger \vec{\nabla} \phi + m^2 \phi^\dagger \phi$$

$$= \pi^\dagger \pi + \vec{\nabla} \phi^\dagger \vec{\nabla} \phi + m^2 \phi^\dagger \phi$$

c) The mode expansion for $\phi(x)$ takes the form

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a(p) e^{-ipx} + b^\dagger(p) e^{ipx} \right) \qquad (2)$$

Make an ansatz for the commutation relations of the modes $a(p)$, $a^\dagger(p)$, $b(p)$, and $b^\dagger(p)$ such that when you explicitly compute the equal time commutator $[\phi(t,\vec{x}), \dot{\phi}^\dagger(t,\vec{y})]$ you obtain the right result for canonical quantisation.

Quantisation is performed by promoting the field operators $\phi(x)$ and $\phi^\dagger(x)$ to operators acting on states in a system's Hilbert space. This process is considered canonical, when the resulting operators fulfill a certain commutation relation, namely

$$[\phi(t,\vec{x}), \dot{\phi}^\dagger(t,\vec{y})] = i\,\delta^{(3)}(\vec{x}-\vec{y}).$$

To achieve this relation, we impose the following commutators for the creators and annihilators

$$[a(\vec{p}), a^\dagger(\vec{q})] = [b(p), b^\dagger(q)] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})$$

$$[a(p), a(q)] = [a^\dagger(p), a^\dagger(q)] = [b(p), b(q)] = [b^\dagger(p), b^\dagger(q)] = \underbrace{\dots}_{\text{other permutations}} = 0$$

Remark: These commutators need not be guessed. They can be computed by carrying over creator and annihilator commutation relations from the real free scalar field and writing $a = \frac{1}{\sqrt{2}}(a_1 + i a_2)$, $b^\dagger = \frac{1}{\sqrt{2}}(a_1^\dagger - i a_2^\dagger)$.

Inserting (2) and the mode expansion for $\dot{\phi}^\dagger(t,\vec{y})$

$$\dot{\phi}^\dagger(t,\vec{y}) = \partial_t \int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_q}}\left(a^\dagger(q)\,e^{iqy} + b(q)\,e^{-iqy}\right)$$

$$= i\int \frac{d^3q}{(2\pi)^3}\sqrt{\frac{E_q}{2}}\left(a^\dagger(q)\,e^{iqy} - b(q)\,e^{-iqy}\right)$$

into the equal-time commutator yields

$$[\phi(t,\vec{x}), \dot{\phi}^\dagger(t,\vec{y})] = i\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\,e^{-ipx}\int\frac{d^3q}{(2\pi)^3}\sqrt{\frac{E_q}{2}}\,e^{iqy}\,[a(p) + b^\dagger(-p), a^\dagger(q) - b(-q)]$$

$$= \frac{i}{2}\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}\sqrt{\frac{E_q}{E_p}}\,e^{i(qy-px)}\Big([a(p), a^\dagger(q)] - \underbrace{[a(p), b(-q)]}_{0} + \overbrace{[b^\dagger(-p), a^\dagger(q)]}^{0} - [b^\dagger(-p), b(-q)]\Big)$$

$$= \frac{i}{2}\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}\sqrt{\frac{E_q}{E_p}}\,e^{i(qy-px)}\left((2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}) + (2\pi)^3\delta^{(3)}(-\vec{p}+\vec{q})\right)$$

$$= i\int\frac{d^3p}{(2\pi)^3}\,e^{ip(y-x)} = i\int\frac{d^3p}{(2\pi)^3}\,e^{iE_p(t-t) - i\vec{p}(\vec{y}-\vec{x})} = i\int\frac{d^3p}{(2\pi)^3}\,e^{ip(\vec{x}-\vec{y})}$$

$$= i\,\delta^{(3)}(\vec{x}-\vec{y})$$

d) Show that the action (1) is invariant under the transformation

$$\phi(x) \longrightarrow e^{i\alpha}\phi(x), \quad \alpha \in \mathbb{R}$$

and derive the associated Noether current and the Noether charge Q in terms of the fundamental fields.

Hint: $Q = \int d^3x\, j^0(x)$ and $j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\,\delta\phi - F^\mu$.

$$S' = \int d^4x \left(\partial^\mu \phi^\dagger(x) \underbrace{e^{-i\alpha}\,\partial_\mu\, e^{i\alpha}}_{\partial_\mu} \phi(x) - m^2 \phi^\dagger(x) \underbrace{e^{-i\alpha}\, e^{i\alpha}}_{1} \phi(x) \right) = S$$

Since $\alpha \in \mathbb{R}$ generates a continuous symmetry, we can Taylor expand the transformed fields in terms of infinitesimal variations of the original field, i.e.

$$\phi(x) \longrightarrow \phi'(x,\alpha) = \phi(x) + \alpha\,\delta\phi(x) + \frac{\alpha^2}{2}\delta^2\phi(x) + \frac{\alpha^3}{3!}\delta^3\phi(x) + \dots$$

$$= \phi(x) + \alpha\,\delta\phi(x) + \sigma(\alpha^2)$$

Since $\phi'(x,\alpha) = e^{i\alpha}\phi(x) = \left(1 + i\alpha + \frac{(i\alpha)^2}{2} + \dots\right)\phi = \phi + \alpha\, i\phi - \frac{\alpha^2}{2}\phi + \dots$, we can deduce $\delta\phi(x) = i\phi(x)$. Similarly for $\phi^\dagger(x)$.

After transformation, the Lagrangian depends on the new fields and therefore on the transformation's parameter $\alpha \in \mathbb{R}$. Therefore, it too can be expanded in α

$$\mathcal{L} \longrightarrow \mathcal{L}' = \mathcal{L} + \alpha\,\delta\mathcal{L} + \frac{\alpha^2}{2}\delta^2\mathcal{L} + \frac{\alpha^3}{3!}\delta^3\mathcal{L} + \dots$$

$$= \mathcal{L} + \alpha\,\delta\mathcal{L} + \sigma(\alpha^2)$$

However, since we already know the Lagrangian as well as the action remains unchanged by the transformation, it follows that

$$\delta^n \mathcal{L} = 0 \quad \forall n \in \mathbb{N}$$

We now have all the necessary parts to assemble the Noether current j^μ associated with this transformation

$$j^\mu = \sum_{d.o.f.} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi - F^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^+)} \delta\phi^+ - F^\mu = (\partial^\mu \phi^+) i\phi - (\partial^\mu \phi) i \phi^+ - F^\mu$$

where F^μ, due to $\partial_\mu F^\mu := \delta\mathcal{L} = 0$, is just a constant which can be set to zero, i.e.

$$j^\mu = i\left(\phi\,\partial^\mu \phi^+ - \phi^+ \partial^\mu \phi\right)$$

The charge then follows as

$$Q = \int d^3x\; j^0(x) = i\int d^3x \left(\phi\,\partial^0 \phi^+ - \phi^+ \partial^0 \phi\right) = i\int d^3x \left(\phi\,\partial_t \phi^+ - \phi^+ \partial_t \phi\right)$$

$$= i\int d^3x \int \frac{d^3p}{(2\pi)^3} \frac{e^{-ipx}}{\sqrt{2E_p}}\, i\int \frac{d^3q}{(2\pi)^3} \sqrt{\frac{E_q}{2}}\, e^{iqx} \Big([a(p) + b^+(p)][a^+(q) - b(-q)] + [a^+(-p) + b(p)][a(-q) - b^+(q)] \Big)$$

$$= -\frac{1}{2}\int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \sqrt{\frac{E_q}{E_p}} \int d^3x\, e^{i(q-p)x} \Big(a(p)\,a^+(q) - a(p)\,b(-q) + b^+(-p)\,a^+(q) - b^+(-p)\,b(-q)$$
$$+ a^+(p)\,a(-q) - a^+(-p)\,b^+(q) + b(p)\,a(-q) - b(p)\,b^+(q) \Big)$$

$$= -\frac{1}{2}\int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \sqrt{\frac{E_q}{E_p}}\, (2\pi)^3 \delta^{(3)}(\vec{q} - \vec{p})\, e^{i(E_q - E_p)t} \left(\dots \right)$$

$$= -\frac{1}{2}\int \frac{d^3p}{(2\pi)^3} \Big(a(p)\,a^+(p) - \underline{a(p)\,b(-p)} + \underline{b^+(-p)\,a^+(p)} - b^+(-p)\,b(-p)$$
$$+ a^+(-p)\,a(-p) - \underline{a^+(-p)\,b^+(p)} + \underline{b(p)\,a(-p)} - b(p)\,b^+(p) \Big)$$

$$= -\frac{1}{2}\int \frac{d^3p}{(2\pi)^3} \Big(a^+(p)\,a(p) + (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p}) - b^+(-p)\,b(-p) + a^+(-p)\,a(-p) - b^+(p)\,b(p) - (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p}) \Big)$$

$$= -\frac{1}{2}\int \frac{d^3p}{(2\pi)^3} \Big(2a^+(p)\,a(p) - 2b^+(p)\,b(p) \Big) = -\int \frac{d^3p}{(2\pi)^3} \Big(a^+(p)\,a(p) - b^+(p)\,b(p) \Big)$$

c) The expression for the Noether charge Q in terms of the modes (in some overall normalisation) reads

$$Q = \int \frac{d^3p}{(2\pi)^3} \Big(a^+(p)\,a(p) - b^+(p)\,b(p) \Big).$$

Compute the charge, i.e. the eigenvalue with respect to Q, of the states $a^+(p)\,|0\rangle$ and $b^+(p)\,|0\rangle$.

Since $a^\dagger(p)|0\rangle$ and $b^\dagger(p)|0\rangle$ are states consisting of just one particle and antiparticle respectively, one would expect their charges to be 1 and -1.

$$Q\, a^\dagger(p)|0\rangle = \int \frac{d^3q}{(2\pi)^3} \left(a^\dagger(q)\, a(q) - b^\dagger(q)\, b(q)\right) a^\dagger(p)|0\rangle$$

$$= \int \frac{d^3q}{(2\pi)^3} \left(a^\dagger(q)\left(a^\dagger(p)\, a(q) + (2\pi)^3 \delta^{(3)}(\vec{q}-\vec{p})\right) - a^\dagger(p)\, b^\dagger(q)\, b(q)\right)|0\rangle$$

$$= \int \frac{d^3q}{(2\pi)^3}\, a^\dagger(q)\, (2\pi)^3 \delta^{(3)}(\vec{q}-\vec{p})\, |0\rangle = +\, a^\dagger(p)|0\rangle$$

$$Q\, b^\dagger(p)|0\rangle = \int \frac{d^3q}{(2\pi)^3} \left(a^\dagger(q)\, a(q) - b^\dagger(q)\, b(q)\right) b^\dagger(p)|0\rangle$$

$$= \int \frac{d^3q}{(2\pi)^3} \left(b^\dagger(p)\, a^\dagger(q)\, a(q) - b^\dagger(q)\left(b^\dagger(p)\, b(q) + (2\pi)^3 \delta^{(3)}(\vec{q}-\vec{p})\right)\right)|0\rangle$$

$$= -\int \frac{d^3q}{(2\pi)^3}\, b^\dagger(q)\, (2\pi)^3 \delta^{(3)}(\vec{q}-\vec{p})\, |0\rangle = -\, b^\dagger(p)|0\rangle,$$

where we used $a(q)|0\rangle = b(q)|0\rangle = 0$ both times in the second-to-last step.

Problem 2 (5 points)

This time consider the real scalar field $\phi(x)$.

a) Show that the time-ordered product $T\phi(x_1)\phi(x_2)$ and the normal-ordered product $:\phi_1(x)\phi(x_2):$ are both symmetric under the interchange of x_1 and x_2.

$$T\,\phi(x_1)\,\phi(x_2) = \left\{\begin{matrix} \phi(x_1)\,\phi(x_2) & \text{if } x_1^0 > x_2^0 \\ \phi(x_2)\,\phi(x_1) & \text{if } x_2^0 > x_1^0 \end{matrix}\right\} = T\,\phi(x_2)\,\phi(x_1)$$

$$:\phi(x_1)\,\phi(x_2): \;=\; :\left[\phi^+(x_1) + \phi^-(x_1)\right]\left[\phi^+(x_2) + \phi^-(x_2)\right]: \;=\; :\phi^+(x_1)\,\phi^+(x_2): + :\phi^+(x_1)\,\phi^-(x_2):$$

$$+ :\phi^-(x_1)\,\phi^+(x_2): + :\phi^-(x_1)\,\phi^-(x_2): = \phi^+(x_1)\,\phi^+(x_2) + \phi^-(x_2)\,\phi^+(x_1)$$

$$+ \phi^-(x_1)\,\phi^+(x_2) + \phi^-(x_1)\,\phi^-(x_2)$$

$$:\phi(x_2)\,\phi(x_1): = \phi^+(x_2)\,\phi^+(x_1) + \phi^-(x_1)\,\phi^+(x_2) + \phi^-(x_2)\,\phi^+(x_1) + \phi^-(x_2)\,\phi^-(x_1) \qquad \square$$

b) Deduce that the Feynman propagator, $D_F(x_1-x_2)$, has the same symmetry property.

$$D_F(x_1-x_2) := \langle 0|T\,\phi(x_1)\,\phi(x_2)|0\rangle \overset{a)}{=} \langle 0|T\,\phi(x_2)\,\phi(x_1)|0\rangle = D_F(x_2-x_1)$$

c) Check Wick's theorem for the case of three scalar fields,

$$T\,\phi(x_1)\,\phi(x_2)\,\phi(x_3) = :\phi(x_1)\,\phi(x_2)\,\phi(x_3): + \phi(x_1)\,D_F(x_2-x_3) + \phi(x_2)\,D_F(x_3-x_1) + \phi(x_3)\,D_F(x_1-x_2). \tag{5}$$

For brevity, instead of $\phi(x_1)$, $\phi(x_2)$, and $\phi(x_3)$, we write ϕ_1, ϕ_2, and ϕ_3.
Now assuming $x_1^0 \geq x_2^0 \geq x_3^0$, we can write

$$\begin{aligned} T\,\phi_1\phi_2\phi_3 &= \phi_1\phi_2\phi_3 = (\phi_1^+ + \phi_1^-)(\phi_2^+ + \phi_2^-)(\phi_3^+ + \phi_3^-) \\ &= \phi_1^+\phi_2^+\phi_3^+ + \phi_1^+\phi_2^+\phi_3^- + \phi_1^+\phi_2^-\phi_3^+ + \phi_1^+\phi_2^-\phi_3^- \\ &\quad + \phi_1^-\phi_2^+\phi_3^+ + \phi_1^-\phi_2^+\phi_3^- + \phi_1^-\phi_2^-\phi_3^+ + \phi_1^-\phi_2^-\phi_3^- \end{aligned}$$

On the other hand, we can compute

$$\begin{aligned} &:\phi_1\phi_2\phi_3: + \phi_1 D_F(x_2-x_3) + \phi_2 D_F(x_3-x_1) + \phi_3 D_F(x_1-x_2) \\ &= \phi_1^+\phi_2^+\phi_3^+ + \phi_3^-\phi_1^+\phi_2^+ + \phi_2^-\phi_1^+\phi_3^+ + \phi_2^-\phi_3^-\phi_1^+ + \phi_1^-\phi_2^+\phi_3^+ + \phi_1^-\phi_3^-\phi_2^+ + \phi_1^-\phi_2^-\phi_3^+ + \phi_1^-\phi_2^-\phi_3^- \\ &\quad + (\phi_1^+ + \phi_1^-)[\phi_2^+, \phi_3^-] + (\phi_2^+ + \phi_2^-)[\phi_1^+, \phi_3^-] + (\phi_3^+ + \phi_3^-)[\phi_1^+, \phi_2^-] \end{aligned}$$

where we used $D_F(x_1-x_2) = \overbrace{\phi(x_1)\,\phi(x_2)} := \begin{cases} [\phi^+(x_1), \phi^-(x_2)] & \text{for } x_1^0 > x_2^0 \\ [\phi^+(x_2), \phi^-(x_1)] & \text{for } x_2^0 > x_1^0 \end{cases}$.
Making use of the fact that the
commutators of free modes $\phi^+(x_i)$ and $\phi^-(x_j)$ are simply scalars, i.e.

$$[\phi^+(x), \phi^-(y)] = \int \frac{d^3p}{(2\pi)^3} \frac{e^{-ipx}}{\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3} \frac{e^{iqy}}{\sqrt{2E_q}} \underbrace{[a(p), a^\dagger(q)]}_{(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})} = \int \frac{d^3p}{(2\pi)^3} \frac{e^{-ip(x-y)}}{2E_p} = D(x-y),$$

we can exchange the order of any free mode with a free mode commutator.

Thus, our expression containing commutators can be expanded further

$$:\phi_1\phi_2\phi_3: + \phi_1 D_F(x_2-x_3) + \phi_2 D_F(x_3-x_1) + \phi_3 D_F(x_1-x_2)$$

$$= \phi_1^+\phi_2^+\phi_3^+ + \phi_3^-\phi_1^+\phi_2^+ + \phi_2^-\phi_1^+\phi_3^+ + \phi_2^-\phi_3^-\phi_1^+ + \phi_1^-\phi_2^+\phi_3^+ + \phi_1^-\phi_3^-\phi_2^+ + \phi_1^-\phi_2^-\phi_3^+ + \phi_1^-\phi_2^-\phi_3^-$$

$$+\phi_1^+[\phi_2^+,\phi_3^-] + \phi_1^-[\phi_2^+,\phi_3^-] + [\phi_1^+,\phi_3^-]\phi_2^+ + \phi_2^-[\phi_1^+,\phi_3^-] + [\phi_1^+,\phi_2^-]\phi_3^+ + [\phi_1^+,\phi_2^-]\phi_3^-$$

$$= \phi_1^+\phi_2^+\phi_3^+ + \phi_3^-\phi_1^+\phi_2^+ + \phi_2^-\phi_1^+\phi_3^+ + \phi_2^-\phi_3^-\phi_1^+ + \phi_1^-\phi_2^+\phi_3^+ + \phi_1^-\phi_3^-\phi_2^+ + \phi_1^-\phi_2^-\phi_3^+ + \phi_1^-\phi_2^-\phi_3^-$$

$$+ \phi_1^+\phi_2^+\phi_3^- - \phi_1^+\phi_3^-\phi_2^+ + \phi_1^-\phi_2^+\phi_3^- - \phi_1^-\phi_3^-\phi_2^+ + \phi_1^+\phi_3^-\phi_2^+ - \phi_3^-\phi_1^+\phi_2^+$$

$$+ \phi_2^-\phi_1^+\phi_3^- - \phi_2^-\phi_3^-\phi_1^+ + \phi_1^+\phi_2^-\phi_3^+ - \phi_2^-\phi_1^+\phi_3^+ + \phi_1^+\phi_2^-\phi_3^- - \phi_2^-\phi_1^+\phi_3^-$$

$$= \phi_1^+\phi_2^+\phi_3^+ + \phi_1^-\phi_2^+\phi_3^+ + \phi_1^-\phi_2^-\phi_3^+ + \phi_1^-\phi_2^-\phi_3^- + \phi_1^+\phi_2^+\phi_3^- + \phi_1^-\phi_2^+\phi_3^- + \phi_1^+\phi_2^-\phi_3^+ + \phi_1^+\phi_2^-\phi_3^-$$

$$= (\phi_1^+ + \phi_1^-)\phi_2^+\phi_3^+ + \phi_1^-\phi_2^-(\phi_3^+ + \phi_3^-) + (\phi_1^+ + \phi_1^-)\phi_2^+\phi_3^- + \phi_1^+\phi_2^-(\phi_3^+ + \phi_3^-)$$

$$= \phi_1\phi_2^+(\phi_3^+ + \phi_3^-) + (\phi_1^+ + \phi_1^-)\phi_2^-\phi_3 = \phi_1\phi_2^+\phi_3 + \phi_1\phi_2^-\phi_3 = \phi_1\phi_2\phi_3 = T\phi_1\phi_2\phi_3$$

Since we have shown in parts a) and b) that both the normal-ordered product and the Feynman propagator are symmetric under the exchange of arguments, this result has to hold even for e.g. $T\phi_1\phi_2\phi_3 = \phi_2\phi_1\phi_3$, i.e. without making the initial assumption $x_1^0 > x_2^0 > x_3^0$.

Now consider an interacting real scalar field theory with Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4.$$

d) Examine $\langle 0|S|0\rangle$ to order λ^2 in this theory and identify the different diagrams that arise from an application of Wick's theorem.

Hint: $\langle 0|S|0\rangle = \langle 0|\,T\exp\left[-i\int_{-\infty}^{\infty} dt\; H_I(t)\right]|0\rangle$.

To expand the expression given as a hint to second order in the coupling constant λ, we first need a relation between $H_I(t)$ and $\mathcal{L}_{int} = -\frac{\lambda}{4!}\phi^4$.

Differing from what one might expect from definitions of interaction picture operators such as

$$\phi_I(t,\vec{x}) := e^{iH_0(t-t_0)}\,\phi(t_0,\vec{x})\,e^{-iH_0(t-t_0)} \quad \text{and} \quad \pi_I(t,\vec{x}) := e^{iH_0(t-t_0)}\,\pi(t_0,\vec{x})\,e^{-iH_0(t-t_0)},$$

$H_I(t)$ is not equal to $e^{iH_0(t-t_0)}H(t_0)e^{-iH_0(t-t_0)}$ but rather

$$H_I(t) := e^{iH_0(t-t_0)}\,H_{int}(t_0)\,e^{-iH_0(t-t_0)}, \quad \text{where} \quad H(t) = H_0(t) + H_{int}(t).$$

Now since $L = L_0 + L_{int}$ and $H = \int d^3x\, \pi(t,\vec{x})\,\dot{\phi}(t,\vec{x}) - L_0 - L_{int} = H_0 + H_{int}$, we can see that $H_{int} = -L_{int}$, i.e. performing a Legendre transformation to move from Lagrangian to Hamiltonian does not introduce new interaction terms as long as the Lagrangian does not contain interactions with time-derivatives of the field. Otherwise

$$\pi(t,\vec{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(t,\vec{x})}$$

would contain higher powers of the field than just the first.
We now have to necessary relation

$$H_I = \int d^3x\, e^{iH_0(t-t_0)}(-\mathcal{L}_{int})e^{-iH_0(t-t_0)} = \int d^3x\, e^{iH_0(t-t_0)}\frac{\lambda}{4!}\phi^4 e^{-iH_0(t-t_0)}$$

$$= \int d^3x \frac{\lambda}{4!} e^{iH_0(t-t_0)}\phi\, e^{-iH_0(t-t_0)} e^{iH_0(t-t_0)}\phi\, e^{-iH_0(t-t_0)} \dots e^{iH_0(t-t_0)}\phi\, e^{-iH_0(t-t_0)} = \int d^3x \frac{\lambda}{4!}\phi^4$$

and can expand $\langle 0|S|0\rangle$ into

$$\langle 0|S|0\rangle = \langle 0|T\,\mathbb{1}|0\rangle + \langle 0|T\left(-i\frac{\lambda}{4!}\int d^4x\,\phi^4\right)|0\rangle + \langle 0|T\,\frac{1}{2}\left(-i\frac{\lambda}{4!}\int d^4x\,\phi^4\right)^2|0\rangle + \mathcal{O}(\lambda^3)$$

$$= 1 - i\frac{\lambda}{4!}\int d^4x\,\langle 0|T\,\phi^4(x)|0\rangle - \frac{\lambda^2}{2\cdot 4!^2}\int d^4x\int d^4y\,\langle 0|T\,\phi^4(x)\,\phi^4(y)|0\rangle + \mathcal{O}(\lambda^3)$$

To find the Feynman diagrams these terms represent, we should be aware that we are looking at vacuum to vacuum processes without external particles. Therefore, when applying Wick's theorem, we need all fields to be contracted. We first look at $\langle 0|T\,\phi^4(x)|0\rangle$

$$\langle 0|T\,\phi^4(x)|0\rangle = \underbrace{\langle 0|:\phi^4(x):|0\rangle}_{0} + \langle 0|\phi(x)\phi(x)\phi(x)\phi(x)|0\rangle$$

$\frac{1}{S} = \frac{1}{8}$

This process can be identified with the following diagram:

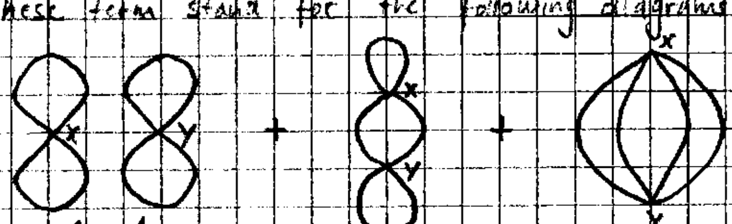
The second order term gives

$$\langle 0|T\,\phi^4(x)\phi^4(y)|0\rangle = \langle 0|:\phi^4(x)\phi^4(y):|0\rangle + \langle 0|\phi(x)\phi(x)\phi(x)\phi(x)\phi(y)\phi(y)\phi(y)\phi(y)|0\rangle$$

$$+ \langle 0|\phi(x)\phi(x)\phi(x)\phi(x)\phi(y)\phi(y)\phi(y)\phi(y)|0\rangle$$

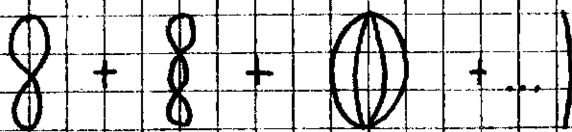
$$+ \langle 0|\phi(x)\phi(x)\phi(x)\phi(x)\phi(y)\phi(y)\phi(y)\phi(y)|0\rangle$$

These term stand for the following diagrams

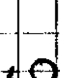
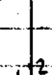


$\frac{1}{S} = \frac{1}{128}$ $\frac{1}{S} = \frac{1}{16}$ $\frac{1}{S} = \frac{1}{48}$

e) Confirm that, to order λ^2, the combinatorial (symmetry) factors work out such that the vacuum-to-vacuum amplitude is given by the exponential sum of distinct vacuum bubble types,

$\langle 0|S|0\rangle = \exp\Big($  $+ \dots \Big)$

Combining all diagrams and their combinatorial factors found in d) into a sum

$\langle 0|S|0\rangle = 1 + (-i\lambda)\Big(\frac{1}{8}$ $\Big) + (-i\lambda)^2\frac{1}{2}\Big(\frac{1}{8}$ $\Big)^2 + (-i\lambda)^2\frac{1}{2}\Big(\frac{1}{8}$ $\Big) + (-i\lambda)^2\frac{1}{2}\Big(\frac{1}{24}$ $\Big) + \mathcal{O}(\lambda)^3$