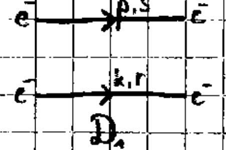
# Model Questions for the Exam by Martin Bies

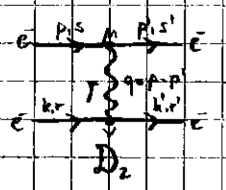
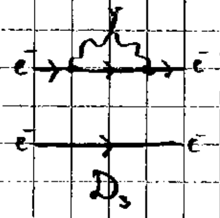
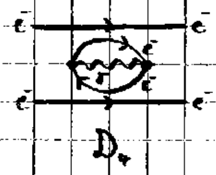
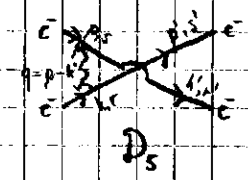
## Exercise 1

Let us consider QED.

a) Draw all diagrams that contribute to $e^- + e^- \to e^- + e^-$ up to order 2. Give interpretations of all diagrams.

$e^0$: 

$e^1$: no diagrams possible with just one vertex for this process

$e^2$:    

$D_1$: no scattering or trivial scattering

$D_2$: real scattering (fully connected, amputated diagram, hence needs to be computed explicitly, is not taken care of implicitly)

$D_3$: charge renormalization

$D_4$: vacuum renormalization

$D_5$: real scattering (another fully connected, amputated diagram)

b) Without justification, state if the following is true:

Be $D$ a Feynman diagram in QED, then $S(D) = 1$.

This statement is true. All symmetry factors in QED are one.

c) Identify the fully connected diagrams in part a). Use the momentum space Feynman rules and your answer to part b) to write down their algebraic expressions.

The full scattering amplitude of a diagram to order n is given by

$$i\mathcal{M}_{fi} = \sum_{k=0}^{n} \sum_{i} \frac{1}{S(D_i)} D_i,$$

where i sums over all fully connected, amputated diagrams at order k.

In the case of $e^- + e^- \longrightarrow e^- + e^-$, we have

$$i\mathcal{M}_{e^-e^-\to e^-e^-} = \frac{1}{S(D_2)} D_2 + \frac{1}{S(D_5)} D_5 \overset{b)}{=} D_2 + D_5 = \bar{u}_{s'}(p')(-ie\gamma^\mu)\, u_s(p) \frac{-i\eta_{\mu\nu}}{(p-p')^2} \bar{u}_{r'}(k')(-ie\gamma^\nu)\, u_r(k)$$

$$- \bar{u}_{r'}(k')(-ie\gamma^\mu)\, u_s(p) \frac{-i\eta_{\mu\nu}}{(p-k')^2} \bar{u}_{s'}(p')(-ie\gamma^\nu)\, u_r(k)$$

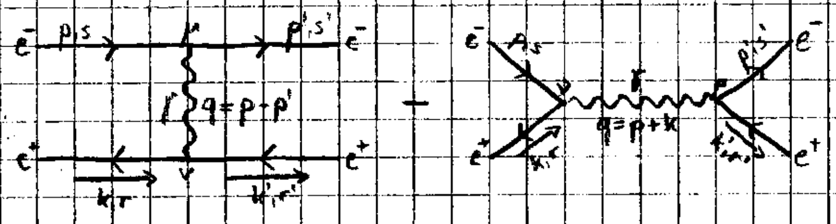
$$= \frac{i}{2} e^2 \left( \bar{u}_{s'}(p')\gamma^\mu u_s(p) \frac{1}{m^2 - pp'} \bar{u}_{r'}(k')\gamma_\mu u_r(k) - \bar{u}_{r'}(k')\gamma^\mu \frac{1}{m^2 - pk'} u_s(p)\gamma_\mu u_r(k) \right)$$

d) What is the amplitude of the process $e^- + e^- \longrightarrow \gamma$ to all orders in perturbation theory? Justify your answer.

The scattering amplitude $i\mathcal{M}_{e^-e^-\to\gamma}$ is zero to every order in a perturbative expansion because it is not permitted by basic physical laws. It violates both momentum and charge conservation.

e) Now consider the process $e^- + e^+ \longrightarrow e^- + e^+$. Draw the fully connected diagrams at order two. Then write down the corresponding algebraic expression.

$i\mathcal{M}_{e^-e^+\to e^-e^+} =$




$$= \bar{u}_{s'}(p')(-ie\gamma^\mu)\, u_s(p) \frac{-i\eta_{\mu\nu}}{(p-p')^2} \bar{v}_r(k)(-ie\gamma^\nu)\, v_{r'}(k')$$

$$- \bar{u}_{s'}(p')(-ie\gamma^\mu)\, v_{r'}(k') \frac{-i\eta_{\mu\nu}}{(p+k)^2} \bar{v}_r(k)(-ie\gamma^\nu)\, u_s(p)$$

$$= \frac{i}{2} e^2 \left( \bar{u}_{s'}(p')\gamma^\mu u_s(p) \frac{1}{m^2 - pp'} \bar{v}_r(k)\gamma_\mu v_{r'}(k') - \bar{u}_{s'}(p')\gamma^\mu v_{r'}(k') \frac{1}{m^2 + pk} \bar{v}_r(k)\gamma_\mu u_s(p) \right)$$

f) Add the contributions from to $e^- + e^+ \longrightarrow e^- + e^+$ from part e) to form $i\mathcal{M}$.
Are there relative minus sign? Justify your answer.
Repeat this task for the results obtained in part c).
There is a relative minus sign in both cases. In d), we get it because two lines cross, hence exchanging the two fermions gives back the other diagram

times a minus sign for the electron exchange. The reasoning in e) is similar except that we also need to employ a rotation of the diagram or a relabelling of time and space axes to convert one diagram into the other.

## Exercise 2

Consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi - g\,\phi\bar{\Psi}\Psi - \frac{\lambda}{4!}\phi^4 + iey^\mu\bar{\Psi}A_\mu\Psi.$$

a) Justify that the following holds for the mass dimensions.

$$[\text{mass}] = [\text{energy}] = [\text{length}^{-1}] = [\text{time}^{-1}]$$

Hint: Recall what units you are working in.

We employ natural units, where we set $\hbar = c = 1$. Under these circumstances, Einstein's famous equation $E = mc^2$ becomes $E = m$ and hence $[E] = [m]$. From $\hbar = 1$, we know that $0 = [\hbar] = [S]$ and can deduce $0 = [S] = [dt\, L]$. Since $L = T - V$, it has to have mass dimension $[L] = [E] = 1$ and therefore $[dt] = [t] = -1$. From $0 = [c] = [v \cdot \lambda] = [\frac{1}{t}] + [d] = 1 + [s]$, we conclude $[d] = -1$.

b) What is the mass dimension of the action $S = \int d^4x\, \mathcal{L}$? What are the mass dimensions of the fields in the Lagrangian? (Assume $[e] = 0$)

As remarked in a), $[S] = 0$. Hence $0 = [S] = [d^4x\,\mathcal{L}] = -4 + [\mathcal{L}]$, $[\mathcal{L}] = 4$.

This results in

$$4 = [\mathcal{L}] = [m^2\phi^2] = 2 + 2[\phi] \implies [\phi] = 1$$

$$4 = [\mathcal{L}] = [m\bar{\Psi}\Psi] = 1 + 2[\Psi] \implies [\Psi] = \frac{3}{2}$$

$$4 = [\mathcal{L}] = [\bar{\Psi}A_\mu\Psi] = 2\cdot\frac{3}{2} + [A_\mu] \implies [A_\mu] = 1$$

c) What representations of Spin(1,3) do the fields $\phi, \Psi$ belong to / transform in? Does $A_\mu$ also transform in a representation of Spin(1,3)? What is the general relation between representations of Spin(1,3) and particles in QFT? Bonus: What representation of U(1) does $A_\mu$ transform in?

The fermionic fields $\Psi$ and $\Psi^\dagger$ transform in the Dirac spinor representation of Spin(1,3).

We know that as a representation of the Lorentz algebra so(1,3)

- the scalar fields $\phi$ and $\phi^\dagger$ transform in what is called the trivial, scalar or fundamental representation

$$R_f(\Lambda) = \mathbb{1} \quad \forall \Lambda$$

- the vector field $A^\mu$ transforms in the vector representation defined as

$$[R_v(\Lambda)]^\mu{}_\nu = \Lambda^\mu{}_\nu \quad \forall \Lambda$$

We can now use that there exists a group homomorphism $\pi$ between the Lorentz algebra so(1,3) and Spin(1,3). Thus, $\phi$ and $A_\mu$ as representations of Spin(1,3) transform under the representation obtained by composing their so(1,3) representations with the group homomorphism $\pi$, i.e.

$$\pi R_f(\Lambda): \mathrm{Spin}(1,3) \longrightarrow so(1,3) \longrightarrow V = \mathbb{R},$$

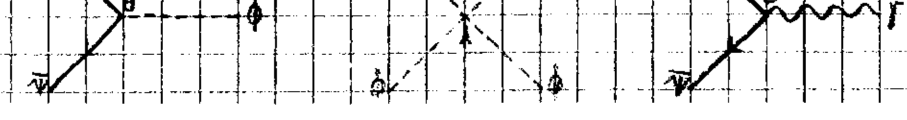
$$\pi [R_v(\Lambda)]^\mu{}_\nu: \mathrm{Spin}(1,3) \longrightarrow so(1,3) \longrightarrow V = \mathbb{R}^{1,3},$$

where $V$ denotes each representation's representation space.

New representations of Spin(1,3) are often found to describe new particles in quantum field theory.

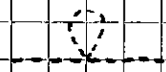
$A_\mu$ transforms in the adjoint representation of U(1).

d) Draw pictures of all interaction vertices in the quantum field theory based on $\mathcal{L}$. Explain your symbols, arrows etc.



Continuous lines indicate fermions, dashed lines scalar particles and curvy lines vector bosons, i.e. photons. Arrows depict negative charge / fermion number flow.

e) Be $\mathcal{D}$ a Feynman diagram in the quantum field theory based on $\mathcal{L}$. Is the following true: $S(\mathcal{D}) = 1$? Prove it or give a counterexample.

Counterexample:  has $S(\mathcal{D}) = 2$.

f) Draw a vacuum bubble that involves all fields $\phi, \psi, \bar{\psi}$, and $A_\mu$.