

Model Questions for the Exam by Martin Bies

Exercise 1

Let us consider QED.

a) Draw all diagrams that contribute to $e^- + e^- \rightarrow e^- + e^-$ up to order 2.

Give interpretations of all diagrams.

$$e^0: \bar{e} \xrightarrow{p,s} \bar{e}$$

$$\bar{e} \xrightarrow{k,r} \bar{e}$$

e^1 : no diagrams possible with just one vertex for this process

$$e^2: \begin{array}{c} \bar{e} \xrightarrow{p,s} \bar{e} \\ \bar{e} \xrightarrow{k,r} \bar{e} \end{array}$$

$$\bar{e} \xrightarrow{\text{loop}} \bar{e}$$

$$\bar{e} \xrightarrow{\text{vacuum}} \bar{e}$$

$$\bar{e} \xrightarrow{\text{box}} \bar{e}$$

D_1 : no scattering or trivial scattering

D_2 : real scattering (fully connected, amputated diagram, hence needs to be computed explicitly, is not taken care of implicitly)

D_3 : charge renormalization

D_4 : vacuum renormalization

D_5 : real scattering (another fully connected, amputated diagram)

b) Without justification, state if the following is true:

Be D a Feynman diagram in QED, then $S(D) = 1$.

This statement is true. All symmetry factors in QED are one.

c) Identify the fully connected diagrams in part a) Use the momentum space Feynman rules and your answer to part b) to write down their algebraic expressions.

The full scattering amplitude of a diagram to order n is given by

$$i\mathcal{M}_f = \sum_{k=0}^n \sum_i \frac{1}{S(D_i)} D_i$$

where i sums over all fully connected, amputated diagrams at order k .

In the case of $e^- + e^- \rightarrow e^- + e^-$ we have

$$\begin{aligned}
 iM_{e^-e^- \rightarrow e^-e^-} &= \frac{1}{s(D)} D_2 + \frac{1}{s(D)} D_3 = D_2 + D_3 = \bar{u}_s(p) (-ie\gamma^\mu) u_s(p) \frac{-i\cancel{D}_{pp}}{(p-p')^2} \bar{u}_s(k) (-ie\gamma^\nu) u_s(k) \\
 &\quad - \bar{u}_s(k) (-ie\gamma^\mu) u_s(p) \frac{-i\cancel{D}_{pp}}{(p-p')^2} \bar{u}_s(p) (-ie\gamma^\nu) u_s(k) \\
 &= \frac{1}{2} e^2 \left(\bar{u}_s(p) \gamma^\mu u_s(p) \frac{1}{m^2 - p^2} \bar{u}_s(k) \gamma_\mu u_s(k) - \bar{u}_s(k) \gamma^\mu u_s(p) \frac{1}{m^2 - p^2} \bar{u}_s(p) \gamma_\mu u_s(k) \right)
 \end{aligned}$$

d) What is the amplitude of the process $e^- + e^- \rightarrow \gamma$ to all orders in perturbation theory? Justify your answer.

The scattering amplitude $iM_{e^-e^- \rightarrow \gamma}$ is zero to every order in a perturbative expansion because it is not permitted by basic physical laws. It violates both momentum and charge conservation.

e) Now consider the process $e^- + e^+ \rightarrow e^- + e^+$. Draw the fully connected diagrams at order two. Then write down the corresponding algebraic expression.

$$\begin{aligned}
 iM_{e^-e^+ \rightarrow e^-e^+} &= \bar{u}_s(p) (-ie\gamma^\mu) u_s(p) \frac{-i\cancel{D}_{pp}}{(p-p')^2} \bar{v}_s(k) (-ie\gamma^\nu) v_s(k) \\
 &\quad - \bar{u}_s(p) (-ie\gamma^\mu) v_s(k) \frac{-i\cancel{D}_{pk}}{(p+k)^2} \bar{v}_s(k) (-ie\gamma^\nu) u_s(p) \\
 &= \frac{1}{2} e^2 \left(\bar{u}_s(p) \gamma^\mu u_s(p) \frac{1}{m^2 - p^2} \bar{v}_s(k) \gamma_\mu v_s(k) - \bar{u}_s(p) \gamma^\mu v_s(k) \frac{1}{m^2 - p^2} \bar{v}_s(k) \gamma_\mu u_s(p) \right)
 \end{aligned}$$

f) Add the contributions from $e^- + e^+ \rightarrow e^- + e^+$ from part c) to form iM .

Are there relative minus signs? Justify your answer.

Repeat this task for the results obtained in part c).

There is a relative minus sign in both cases. In d), we get it because two lines cross, hence exchanging the two fermions gives back the other diagram.

times a minus sign for the electron exchange. The reasoning in c) is similar except that we also need to employ a rotation of the diagram or a relabelling of time and space axes to convert one diagram into the other.

Exercise 2

Consider the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 + \bar{\psi} (\gamma^\mu \partial_\mu - m) \psi - g \phi \bar{\psi} \psi - \frac{\lambda}{4!} \phi^4 + i c \gamma^\mu \bar{\psi} \lambda_\mu \psi.$$

a) Justify that the following holds for the mass dimensions

$$[\text{mass}] = [\text{energy}] = [\text{length}] = [\text{time}^{-1}]$$

Hint: Recall what units you are working in.

We employ natural units, where we set $\hbar = c = 1$. Under these circumstances,

Einstein's famous equation $E = mc^2$ becomes $E = m$ and hence $[E] = [m]$.

From $\hbar = 1$, we know that $0 = [\hbar] = [S]$ and can deduce $0 = [S] = [d + L]$.

Since $L = T - V$, it has to have mass dimension $[L] = [E] = 1$ and therefore

$[d] = [t] = -1$. From $0 = [c] = [v \cdot \lambda] = [\frac{1}{t}] + [d] = 1 + [d]$, we conclude $[d] = -1$.

b) What is the mass dimension of the action $S = \int d^4x \mathcal{L}$? What are the mass dimensions of the fields in the Lagrangian? (Assume $[c] = 0$)

As remarked in a), $[S] = 0$. Hence $0 = [S] = [d^4x \mathcal{L}] = -4 + [\mathcal{L}]$, $[\mathcal{L}] = 4$.

This results in

$$4 = [\mathcal{L}] = [m^2 \phi^2] = 2 + 2[\phi] \Rightarrow [\phi] = 1$$

$$4 = [\mathcal{L}] = [m \bar{\psi} \psi] = 1 + 2[\psi] \Rightarrow [\psi] = \frac{3}{2}$$

$$4 = [\mathcal{L}] = [\bar{\psi} \lambda_\mu \psi] = 2 \cdot \frac{3}{2} + [\lambda_\mu] \Rightarrow [\lambda_\mu] = 1$$

c) What representations of $\text{Spin}(1,3)$ do the fields ϕ, ψ belong to / transform in? Does λ_μ also transform in a representation of $\text{Spin}(1,3)$? What

is the general relation between representations of $\text{Spin}(1,3)$ and particles in QFT? Bonus: What representation of $U(1)$ does A_μ transform in?

The fermionic fields ψ and $\bar{\psi}$ transform in the Dirac spinor representation of $\text{Spin}(1,3)$.

We know that as a representation of the Lorentz algebra $\mathfrak{so}(1,3)$

- the scalar fields ϕ and $\phi^\dagger$ transform in what is called the trivial, scalar or fundamental representation

$$R_\Lambda(\Lambda) = 1 \quad \forall \Lambda$$

- the vector fields A^μ transform in the vector representation defined as

$$[R_\Lambda(\Lambda)]^\mu_\nu = \Lambda^\mu_\nu \quad \forall \Lambda$$

We can now use that there exists a group homomorphism π between the Lorentz algebra $\mathfrak{so}(1,3)$ and $\text{Spin}(1,3)$. Thus, ϕ and A_μ as representations of $\text{Spin}(1,3)$ transform under the representation obtained by composing their $\mathfrak{so}(1,3)$ representations with the group homomorphism π ; i.e.

$$\pi R_\Lambda(\Lambda): \text{Spin}(1,3) \longrightarrow \mathfrak{so}(1,3) \longrightarrow V = \mathbb{R},$$

$$\pi [R_\Lambda(\Lambda)]^\mu_\nu: \text{Spin}(1,3) \longrightarrow \mathfrak{so}(1,3) \longrightarrow V = \mathbb{R}^{1,3},$$

where V denotes each representation's representation space.

New representations of $\text{Spin}(1,3)$ are often found to describe new particles in quantum field theory.

A_μ transforms in the adjoint representation of $U(1)$.

d) Draw pictures of all interaction vertices in the quantum field theory based on

$\mathcal{L}$. Explain your symbols, arrows etc.



Continuous lines indicate fermions, dashed lines scalar particles and curly lines vector bosons, i.e. photons. Arrows depict negative charge / fermion number flow.

e) Be D a Feynman diagram in the quantum field theory based on $\mathcal{L}$.

Is the following true: $S(D)=1$? Prove it or give a counterexample.

Counterexample:  has $S(D)=2$.

f) Draw a vacuum bubble that involves all fields ϕ , ψ , $\bar{\psi}$, and A_μ .

