Model Questions for the Exam

Quantum Field Theory I, by Martin Bies

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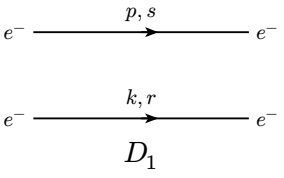
Exercise 1

Let us consider QED.

a) Draw all diagrams that contribute to $e^- + e^- \to e^- + e^-$ up to order 2. Give interpretations of all diagrams.

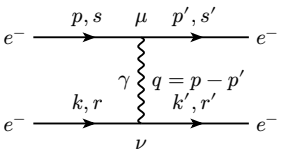
e^0



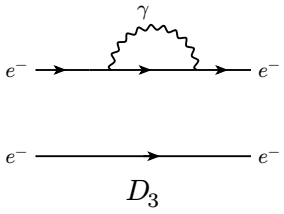
D_1

e^1: no diagrams possible with just one vertex for this process

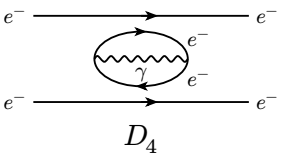
e^2



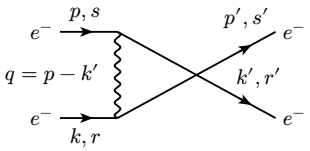
D_2



D_3



D_4



D_5

- $\boldsymbol{D_1}$ no scattering or trivial scattering
- $\boldsymbol{D_2}$ real scattering (fully connected, amputated diagram, hence needs to be computed explicitly, is not taken care of implicitly)
- $\boldsymbol{D_3}$ electron self-energy on an external leg (mass and field-strength renormalization)
- $\boldsymbol{D_4}$ vacuum renormalization
- $\boldsymbol{D_5}$ real scattering (another fully connected, amputated diagram)

b) Without justification, state if the following is true: Be D a Feynman diagram in QED, then $S(D) = 1$.

This statement is true. All symmetry factors in QED are one.

c) Identify the fully connected diagrams in part a). Use the momentum space Feynman rules and your answer to part b) to write down their algebraic expressions.

The full scattering amplitude of a diagram to order n is given by

$$i\mathcal{M}_f = \sum_{k=0}^{n} \sum_i \frac{1}{S(D_i)} D_i,$$

where i sums over all fully connected, amputated diagrams at order k. In the case of $e^- + e^- \to e^- + e^-$ we have

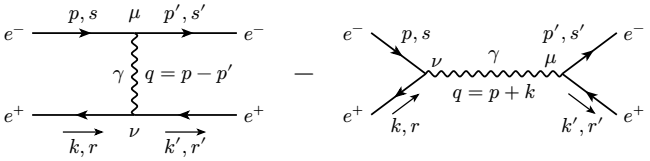
$$
\begin{aligned}
i\mathcal{M}_{e^-e^-\to e^-e^-} &= \frac{1}{S(D_2)} D_2 + \frac{1}{S(D_5)} D_5 \overset{\text{b)}}{=} D_2 + D_5 \\
&= \bar{u}_{s'}(p')(-ie\gamma^\mu)u_s(p) \frac{-i\eta_{\mu\nu}}{(p-p')^2} \bar{u}_{r'}(k')(-ie\gamma^\nu)u_r(k) \\
&\quad -\bar{u}_{r'}(k')(-ie\gamma^\mu)u_s(p) \frac{-i\eta_{\mu\nu}}{(p-k')^2} \bar{u}_{s'}(p')(-ie\gamma^\nu)u_r(k) \\
&= \frac{i}{2} e^2 \left(\bar{u}_{s'}(p')\gamma^\mu u_s(p) \frac{1}{m^2 - pp'} \bar{u}_{r'}(k')\gamma_\mu u_r(k) - \bar{u}_{r'}(k')\gamma^\mu u_s(p) \frac{1}{m^2 - pk'} \bar{u}_{s'}(p')\gamma_\mu u_r(k) \right).
\end{aligned}
$$

d) What is the amplitude of the process $e^- + e^- \to \gamma$ to all orders in perturbation theory? Justify your answer.

The scattering amplitude $i\mathcal{M}_{e^-e^-\to\gamma}$ is zero to every order in a perturbative expansion because it is not permitted by basic physical laws. It violates both momentum and charge conservation.

e) Now consider the process $e^- + e^+ \to e^- + e^+$. Draw the fully connected diagrams at order two. Then write down the corresponding algebraic expression.

$i\mathcal{M}_{e^-e^+\to e^-e^+} =$



$$
\begin{aligned}
&= \bar{u}_{s'}(p')(-ie\gamma^\mu)u_s(p) \frac{-i\eta_{\mu\nu}}{(p-p')^2} \bar{v}_r(k)(-ie\gamma^\nu)v_{r'}(k') \\
&\quad -\bar{u}_{s'}(p')(-ie\gamma^\mu)v_{r'}(k') \frac{-i\eta_{\mu\nu}}{(p+k)^2} \bar{v}_r(k)(-ie\gamma^\nu)u_s(p) \\
&= \frac{i}{2} e^2 \left(\bar{u}_{s'}(p')\gamma^\mu u_s(p) \frac{1}{m^2 - pp'} \bar{v}_r(k)\gamma_\mu v_{r'}(k') - \bar{u}_{s'}(p')\gamma^\mu v_{r'}(k') \frac{1}{m^2 + pk} \bar{v}_r(k)\gamma_\mu u_s(p) \right).
\end{aligned}
$$

f) Add the contributions to $e^- + e^+ \to e^- + e^+$ from part e) to form $i\mathcal{M}$. Are there relative minus signs? Justify your answer. Repeat this task for the results obtained in part c).

There is a relative minus sign in both cases. In c), we get it because two lines cross, hence exchanging the two fermions gives back the other diagram times a minus sign for the electron exchange. The reasoning in e) is similar except that we also need to employ a rotation of the diagram or a relabelling of time and space axes to convert one diagram into the other.

Exercise 2

Consider the Lagrangian

$$
\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 + \bar{\Psi}\big(i\gamma^\mu\partial_\mu - m\big)\Psi - g\varphi\bar{\Psi}\Psi - \frac{\lambda}{4!}\varphi^4 + ie\gamma^\mu\bar{\Psi}A_\mu\Psi.
$$

a) Justify that the following holds for the mass dimensions:

$$
[\text{mass}] = [\text{energy}] = [\text{length}^{-1}] = [\text{time}^{-1}].
$$

Hint: Recall what units you are working in.

We employ natural units, where we set $\hbar = c = 1$. Under these circumstances, Einstein's famous equation $E = mc^2$ becomes $E = m$ and hence $[E] = [m]$. From $\hbar = 1$, we know that $0 = [\hbar] = [S]$ and can deduce $0 = [S] = [\mathrm{d}t\, L]$. Since $L = T - V$, it has to have mass dimension $[L] = [E] = 1$ and therefore $[\mathrm{d}t] = [t] = -1$. From $0 = [c] = [\nu \cdot \lambda] = \left[\frac{1}{t}\right] + [d] = 1 + [d]$, we conclude $[d] = -1$.

b) What is the mass dimension of the action $S = \int \mathrm{d}^4x\, \mathcal{L}$? What are the mass dimensions of the fields in the Lagrangian? (Assume $[e] = 0$.)

As remarked in a), $[S] = 0$. Hence $0 = [S] = \left[\mathrm{d}^4x\, \mathcal{L}\right] = -4 + [\mathcal{L}]$, $[\mathcal{L}] = 4$. This results in

$$\begin{aligned} 4 = [\mathcal{L}] = \left[m^2\varphi^2\right] = 2 + 2[\varphi] \quad &\Longrightarrow \quad [\varphi] = 1, \\ 4 = [\mathcal{L}] = \left[m\bar{\Psi}\Psi\right] = 1 + 2[\Psi] \quad &\Longrightarrow \quad [\Psi] = \frac{3}{2}, \\ 4 = [\mathcal{L}] = \left[\bar{\Psi}A_\mu\Psi\right] = 2 \cdot \frac{3}{2} + \left[A_\mu\right] \quad &\Longrightarrow \quad \left[A_\mu\right] = 1. \end{aligned}$$

c) What representations of $\mathrm{Spin}(1,3)$ do the fields φ, Ψ belong to / transform in? Does A_μ also transform in a representation of $\mathrm{Spin}(1,3)$? What is the general relation between representations of $\mathrm{Spin}(1,3)$ and particles in QFT? *Bonus:* What representation of $U(1)$ does A_μ transform in?

The fermionic fields Ψ and $\Psi^\dagger$ transform in the Dirac spinor representation of $\mathrm{Spin}(1,3)$.

We know that as a representation of the Lorentz algebra $\mathfrak{so}(1,3)$

- the scalar fields φ and $\varphi^\dagger$ transform in what is called the trivial or scalar representation
$$R_f(\Lambda) = \mathbb{1} \quad \forall \Lambda,$$
- the vector field A^μ transforms in the vector representation defined as
$$[R_v(\Lambda)]^\mu_{\ \nu} = \Lambda^\mu_{\ \nu} \quad \forall \Lambda.$$

We can now use that there exists a group homomorphism $\pi : \mathrm{Spin}(1,3) \to \mathrm{SO}(1,3)^+$, the double cover of the (proper orthochronous) Lorentz group. Thus, φ and A_μ as representations of $\mathrm{Spin}(1,3)$ transform under the representation obtained by composing their Lorentz group representations with π, i.e.

$$\begin{aligned} R_f \circ \pi : \mathrm{Spin}(1,3) \longrightarrow \mathrm{SO}(1,3)^+ \longrightarrow \mathrm{GL}(V), \quad V = \mathbb{R}, \\ R_v \circ \pi : \mathrm{Spin}(1,3) \longrightarrow \mathrm{SO}(1,3)^+ \longrightarrow \mathrm{GL}(V), \quad V = \mathbb{R}^{1,3}, \end{aligned}$$

where V denotes each representation's representation space.

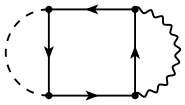
New representations of $\mathrm{Spin}(1,3)$ are often found to describe new particles in quantum field theory.

A_μ transforms in the adjoint representation of $U(1)$.

d) Draw pictures of all interaction vertices in the quantum field theory based on $\mathcal{L}$. Explain your symbols, arrows etc.

Continuous lines indicate fermions, dashed lines scalar particles and curvy lines vector bosons, i.e. photons. Arrows depict negative charge / fermion number flow.

e) Be D a Feynman diagram in the quantum field theory based on $\mathcal{L}$. Is the following true: $S(D) = 1$? Prove it or give a counterexample.

Counterexample: has $S(D) = 2$.

f) Draw a vacuum bubble that involves all fields φ, Ψ, $\bar{\Psi}$ and A_μ.