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Heidelberg University
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General Relativity (MKTP3) Summer Term 2015

Exercise sheet 2 **21 April 2015**

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1. (10 points) **Lorentz transformations and group theory**
We've seen in the lecture that four-vectors that e.g. denote events in spacetime are denoted with greek indices, x^μ, where $\mu = 0, 1, 2, 3$:

$$x^\mu = \begin{pmatrix} ct \\ x^i \end{pmatrix}.$$

Scalar products are defined as

$$x \cdot y := x_\mu y^\mu = \eta_{\mu\nu} x^\mu y^\mu,$$

therefore, the norm of a vector x^μ is

$$|x|^2 = x \cdot x = \eta_{\mu\nu} x^\mu x^\nu.$$

We call $\eta_{\mu\nu}$ the *Minkowski metric*;

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$

As we have learned from the lecture, Lorentz transformations are *linear*, that means we can write event x^μ a new coordinate frame as

$$x'^\rho = \Lambda^\rho_\mu x^\mu.$$

(a) By requiring that $|x'|^2 = |x|^2$, how does $\eta_{\mu\nu}$ transform?

(b) Let's see if Lorentz boosts form a so-called group. A group is defined as a set G that has an operation $\langle a, b\rangle$ for $a, b \in G$ with the following conditions:

- $\langle a, b\rangle \in G$ (i.e. the operation doesn't allow you to "leave" the set)
- $\langle\langle a, b\rangle, c\rangle = \langle a, \langle b, c\rangle\rangle$ (associativity)
- $\exists! e \in G : \langle e, a\rangle = a$ (i.e. there is an element called identity e that leaves a as it is under the operation)
- $\forall x \exists! a^{-1} : \langle a, a^{-1}\rangle = e$ (i.e. there is an inverse for each element, which yields the identity under $\langle\cdot,\cdot\rangle$.)

Show that the simplest Lorentz boosts, i.e. along one axis (here x),

$$\Lambda_\mu^\nu(\psi) = \begin{pmatrix} \cosh\psi & -\sinh\psi & 0 & 0 \\ -\sinh\psi & \cosh\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

with the rapidity

$$\psi = \operatorname{artanh}\frac{v}{c}$$

fulfills the requirements for a group, the operation being simple matrix multiplication. For simplicity, don't worry about the uniqueness of the inverse a^{-1} and the identity e.

(c) A group is called abelian if $\langle a, b\rangle = \langle b, a\rangle$. Is the group of Lorentz-boosts abelian?
Extra question: Is the group of rotations in $\mathbb{R}^3$ abelian? (think Euler-angles)

(d) Write down Λ in the form

$$\Lambda = \exp(\psi L).$$

L is called the generator of the group. What is L for the Lorentz-transform along the x-axis?

2. (10 points) **Journey to Kepler-438b**
Kepler-438b is the catchy name that astronomers chose for one of the most earth-like exoplanets that has been found to date. Its host star lies in the constellation of Lyra, about $d = 145\,\mathrm{pc} = 470\,\mathrm{ly}$ from our own solar system.

(a) If we had a spaceship that could do the journey there, at what speed would it need to travel in order to arrive in 470 years, as the crew experience it (i.e. in their proper time)? How about 47 years?

(b) Imagine the ship was travelling at $v = 0.9c$. What speed would the ship travel at if the captain ordered the ship to increase its velocity by $u = 0.1c$?

(c) If the ship was of length L in their comoving frame, what length L' would a resting observer measure?

(d) Their communication with home was decided to be a on a specific electromagnetic frequency f. What frequency f' do they need to tune into in order to recieve signals from the (resting) sender?

3. (10 points) **Boosting the field tensor**
The electromagnetic field tensor $F^{\mu\nu}$ can be written as

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & -B_3 & B_2 \\ E_2/c & B_3 & 0 & -B_1 \\ E_3/c & -B_2 & B_1 & 0 \end{pmatrix},$$

with the electric and magnetic field components E_i and B_i.

(a) Calculate $F_{\mu\nu}$.

(b) Boost $F^{\mu\nu}$ with the Lorentz transform Λ_μ^ν from exercise 1 to get $F'^{\rho\sigma} = \Lambda_\mu^\rho \Lambda_\nu^\sigma F^{\mu\nu}$.

(c) What are the new electric and magnetic fields E_i' and B_i'?

(d) Calculate $F_{\mu\nu}F^{\mu\nu}$ and $F'_{\rho\sigma}F'^{\rho\sigma}$. By how much do they differ?

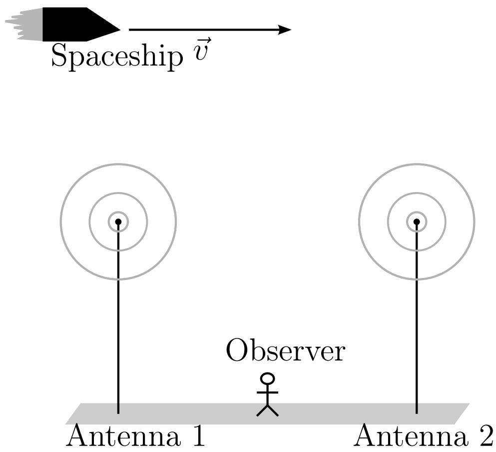
4. (10 points) **Electronics in moving frames**
Consider a rectangular capacitor with capacity $C = \epsilon_0 \epsilon_r \frac{a^2}{d}$. It is now moving along one of its plate-axes with $v > 0$.

(a) Does the capacity change? By what amount?

(b) Does the electric field $E = \frac{U}{d}$ change? By what amount?

(c) Write down the Lorentz force $\vec{F} = q(\vec{E} + (\vec{v} \times \vec{B}))$ of a resting particle between the plates at $d/2$ with charge q with respect to both the resting capacitor and the moving one. How could you reconcile the two forces?

5. (5 points) **Extra: Synchronicity**



Consider two antennae 1 and 2 that are a certain distance L apart. Each recieves a signal from an observer, who's right in the middle of antennas, and responds by sending a signal back to the observer. For him, both antennas responded at the same time.
Now the whole arrangement is seen from a spaceship flying past at a veloicty v from antenna 1 to antenna 2. The crew note that the antennas respond at differing times. Who is right, the observer on the ground or the flight crew?

"Le temps et l'espace... Ce n'est pas la nature qui nous les impose, c'est nous qui les imposons à la nature parce que nous les trouvons commodes."
– Henri Poincaré