ASSIGNMENT 2

Due: Week beginning 27.04.2015.

Problem 2.1 (Path integral and time-ordering):
Show, by using the steps of the derivation of the path integral presented in the lecture, that

$$\int_{q(t_I)\equiv q_I}^{q(t_F)\equiv q_F} \mathcal{D}q\mathcal{D}p\, q(t_1)\, q(t_2)\, e^{i\,S[q,p]} = \langle q_F, t_F|\, T\mathbf{q}_H(t_1)\mathbf{q}_H(t_2)\,|q_I, t_I\rangle \;. \tag{1}$$

Put special emphasis on showing how the time-ordering appears.

Problem 2.2 (Quantum mechanical oscillator and path integral):
The transition amplitude between a state $|q_a,\rangle$ at $t = 0$ and $|q_b\rangle$ at $t = T$ can be schematically expressed as

$$\langle q_b|\, e^{-i\,H\,t}\,|q_a\rangle = \int_{q(0)\equiv q_a}^{q(T)\equiv q_b} \mathcal{D}q\, e^{i\,S[p]} \tag{2}$$

where the integral is done over all the possible trajectories connecting the points $q(0) \equiv q_a$ and $q(T) \equiv q_b$ and $S[q] = \int_0^T dt\,\mathcal{L}[q, \dot{q}]$.

In the case of free fields, $H = \frac{1}{2m}\mathbf{p}^2$, the transition amplitude is simple to calculate. Show that

$$\langle q_b|\, e^{-i\,\frac{1}{2m}\mathbf{p}^2\,T}\,|q_a\rangle = \int dp\,\langle q_b|p\rangle\; e^{-i\,\frac{1}{2m}p^2\,T}\,\langle p|q_a\rangle =$$

$$= \int \frac{dp}{2\pi}\, e^{-i\,p(q_b-q_a)}\, e^{-i\,\frac{1}{2m}p^2\,T} = \sqrt{\frac{m}{2\pi\,i\,T}}\, e^{i\,\frac{m}{2T}(q_a-q_b)^2} \,.$$

We are now going to determine, in a few steps, the amplitude in the case of an harmonic oscillator. The system is characterised by a Lagrangian density

$$\mathcal{L} = \frac{m}{2}\dot{q}^2 - \frac{m\,\omega^2}{2}q^2 \,.$$

a) Recall that the classical trajectory $q_c(t)$ is found by minimising the action:

$$-\frac{\delta S}{\delta q(t)} = m\,\ddot{q}(t) + m\,\omega^2 q(t) = 0 \tag{3}$$

and imposing the boundary conditions $q(0) = q_a$ and $q(T) = q_b$. An arbitrary trajectory $q(t)$ can then be decomposed as $q(t) \equiv q_c(t)+y(t)$ with the boundary conditions $y(0) = y(T) = 0$.

b) Convince yourself that the following expression for the action is exact

$$S[q] = S[q_c] + \int_0^T dt\, \frac{\delta S[q]}{\delta q(t)}\Big|_{q(t)=q_c(t)} y(t) + \frac{1}{2}\int_0^T\!\!\int_0^T dt\,dt'\, \frac{\delta^2 S[q]}{\delta q(t)\delta q(t')}\Big|_{q(t)=q_c(t)} y(t)\,y(t') \,. \tag{4}$$

Then show that we can write

$$\frac{\delta^2 S[q]}{\delta q(t)\delta q(t')} = -m\frac{d^2}{dt^2}\delta(t-t') - m\,\omega^2\delta(t-t') \tag{5}$$

so that

$$S[q] = S[q_c] + \frac{m}{2}\int_0^T dt\,\left(\dot{y}^2(t) - \omega^2 y^2(t)\right) =: S[q_c] + S[y]\,. \tag{6}$$

Then our initial amplitude reads

$$\int \mathcal{D}\,e^{i\,S[q]} = e^{i\,S[q_c]}\int \mathcal{D}y\,e^{i\,S[y]}\,. \tag{7}$$

c) Show that

$$S[q_c] = \frac{m\,\omega}{2\,\sin(\omega\,T)}\left((q_b^2+q_a^2)\cos(\omega\,T) - 2\,q_a q_b\right)\,. \tag{8}$$

Hint: T is not connected to ω by a relation like $T \sim 1/2\pi\omega$.

d) At this point it is convenient to introduce functions ($\mathcal{C}_n$ is a constant to determine)

$$y_n(t) = \mathcal{C}_n \sin\left(\frac{n\,\pi\,t}{T}\right) \tag{9}$$

such that they are orthonormal on the interval $[0,T]$: $\int_0^T dt\,y_n(t)y_m(t) = \delta_{nm}$.
These can be used as a basis to expand any function $y(t)$, satisfying our boundary conditions, as

$$y(t) = \sum_{n=1}^{\infty} a_n y_n(t) \tag{10}$$

by means of a set of constants a_n. Then show that

$$S[y] = \frac{m}{2}\sum_{n=1}^{\infty}\lambda_n\,a_n^2 \tag{11}$$

and determine the constant quantities λ_n.

e) The integral measure can be expressed as (accept it as a postulate, but try to think about it):

$$\mathcal{D}y = J\prod_{n=1}^{\infty} da_n\,, \tag{12}$$

for some constant J. Knowing this, show that

$$F_\omega(T) := \int \mathcal{D}y\,e^{i\,S[y]} = J\prod_{n=1}^{\infty}\sqrt{\frac{2\,\pi\,i}{m\,\lambda_n}}\,. \tag{13}$$

f) We know the exact value of $F_\omega(T)$ for the case of free fields, $\omega = 0$: recall indeed that in this case $F_0(T) = \sqrt{\frac{m}{2\,\pi\,i\,T}}$. On the other hand, one can also calculate $F_0(T)$ by the same procedure we developed until now: show that the λ_n coefficients, when $\omega = 0$, read $\lambda_n^{(0)} = \frac{n^2\pi^2}{T^2}$.

g) Then we can write

$$\frac{F_\omega(T)}{F_0(T)} = \prod_{n=1}^{\infty} \sqrt{\frac{\lambda_n^{(0)}}{\lambda_n}} = \prod_{n=1}^{\infty} \left(1 - \frac{\omega^2 T^2}{\pi^2 n^2}\right)^{-\frac{1}{2}} . \tag{14}$$

Deduce from this that

$$F_\omega(T) = \sqrt{\frac{m\,\omega}{2\pi\, i\, \sin(\omega\, T)}} \tag{15}$$

and, finally, collect everything and write up the result for the transition amplitude!