Heidelberg University
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General Relativity (MKTP3) Summer Term 2015

Exercise sheet 3 **27 April 2015**

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1. (10 points) **“Fun” with indices**
   Consider the funky metric

   $$g^{\mu\nu} = \mathrm{diag}(a, b, -b^{-1}, \tan(c))$$

   (a) Calculate $g_{\mu\nu}$.

   (b) Let's assume that we're in Cartesian coordinates (t, x, y, z). What is the line element $\mathrm{d}s^2$?

   (c) Given

   $$x^\mu = (1, 0, 1, 0)^{\mathrm{t}},$$

   what is x_μ?

   (d) You're given the line element

   $$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}r^2 - r^2\mathrm{d}\phi^2.$$

   What sort of coordinate system are we using? What does the metric look like?

2. (10 points) **Metric tensor and equations of motion**
   The metric can be written as

   $$g_{\mu\nu} = \eta_{\rho\sigma}\frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu},$$

   In 3D Euclidian space, this just boils down to a coordinate transform.

   (a) Find $g_{\mu\nu}$ for polar coordinates (r, θ, ϕ). Neglect the 0-components ($\mu = i = \{1, 2, 3\}$).

   (b) Now limit yourself to the surface of a sphere in $\mathbb{R}^3$, i.e. a 2–sphere. Calculate all remaining Christoffel symbols $\Gamma^\alpha_{\mu\nu}$,

   $$2\Gamma^\alpha_{\mu\nu} = g^{\alpha\rho}(g_{\nu\rho,\mu} + g_{\mu\rho,\nu} - g_{\mu\nu,\rho}).$$

(c) Write down the equations of motion by using the geodesic equation

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0. \tag{1}$$

3. (10 points) **Weak field I: Equations of motion**
Consider the weak–field limit of general relativity with a central potential $\Phi(r) = -\frac{1}{c^2}\frac{GM}{r}$. The line element reads

$$ds^2 = (1 + 2\Phi)\, dt^2 - (1 - 2\Phi(r))\left(dr^2 + r^2 \mathrm{d}\Omega^2\right),$$

with $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2$.

(a) Find the action $S = -mc\int \mathrm{d}\tau \sqrt{\dot{x}_\mu \dot{x}^\mu}$.

(b) You can now read off a Lagrangian L. Why could it also be given as

$$L = (1 + 2\Phi(r))\dot{t}^2 - (1 - 2\Phi(r))\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\ \dot{\phi}^2\right)?$$

(c) Compute the equations of motion from the Euler-Lagrange equations. (Tip: Only keep terms that are linear in Φ or quadratic in $\dot{q}_i$.)

(d) By identifying your equations of motion with the geodesic equation (equation 1), find all non-trivial Christoffel symbols $\Gamma^\alpha_{\mu\nu}$.

4. (10 points) **Weak field II: Let's lens like it's 1999**
We've seen that in a weak–field limit, the metric tensor can be written as

$$g_{\mu\nu} = \begin{pmatrix} 1 + 2\Phi/c^2 & 0 & 0 & 0 \\ 0 & -(1 - 2\Phi/c^2) & 0 & 0 \\ 0 & 0 & -(1 - 2\Phi/c^2) & 0 \\ 0 & 0 & 0 & -(1 - 2\Phi/c^2) \end{pmatrix}, \tag{2}$$

where Φ is the Newtonian gravitational potential.
From this, and from Fermat's principle, we're going to derive the *correct* gravitational lensing deflection angle after having seen the Newtonian case in practice sheet 1.

(a) What are the assumptions that go into finding this limit? Write down the line element $ds^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$ in cartesian coordinates (t, x, y, z).

(b) What special sort of geodesics do light rays follow, i.e. what value does $\mathrm{d}s$ have for light?

(c) From part (b) you can infer the effective velocity of the light ray in the gravitational field,

$$c' = \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}.$$

Approximate it, using that for $a \ll 1$,

$$\sqrt{\frac{1+a}{1-a}} \approx (1+a).$$

The refractive index,

$$n = \frac{c}{c'},$$

can now be written down.
Tip: Use $(1+a)^{-1} \approx (1-a)$.

(d) Fermat's principle states that light takes an optically extremal path, i.e.

$$\delta \int \mathrm{d}l \, n(\vec{x}(l)) = 0.$$

We can write

$$\mathrm{d}l = \left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}\right| \mathrm{d}\lambda = |\dot{\vec{x}}| \, \mathrm{d}\lambda,$$

where we defined $\dot{\vec{x}} := \mathrm{d}\vec{x}/\mathrm{d}\lambda$. Put this into Fermat's principle, which should look like the variation of an action to you now, as in

$$\delta \int \mathrm{d}t \, L(\vec{x}, \dot{\vec{x}}, t) = 0.$$

What is the "Lagrangian" $L(\vec{x}, \dot{\vec{x}}, \lambda)$ in our case?

(e) Show that applying the Euler–Lagrange–Equations,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial \dot{\vec{x}}} = \frac{\partial L}{\partial \vec{x}},$$

yields something of the form

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(n\vec{e}_{\dot{x}}) = (\vec{\nabla} n)|\dot{\vec{x}}|, \tag{3}$$

where $\vec{\nabla} n = \frac{\partial}{\partial \vec{x}} n$ and $\vec{e}_{\dot{x}}$ is the unit vector pointing in $\dot{\vec{x}}$-direction.
We can set $|\dot{\vec{x}}| = 1$ for simplicity because the scale of λ is arbitrary, so let's not worry about that any more.

(f) Simplify equation 3 to the form of

$$n\dot{\vec{e}}_{\dot{x}} = \vec{\nabla} n - \vec{e}_{\dot{x}}(\vec{\nabla} n \cdot \vec{e}_{\dot{x}}).$$

(Tip: $\frac{\mathrm{d}}{\mathrm{d}\lambda} n = \frac{\partial n}{\partial \vec{x}} \frac{\partial \vec{x}}{\partial \lambda}$. Also use that $|\dot{\vec{x}}| = 1$, thus $\dot{\vec{x}} = \vec{e}_{\dot{x}}$)

(g) Since the right–hand–side is subtracting the light–ray–direction of $\vec{\nabla} n$ from $\vec{\nabla} n$, the only remaining part is the one perpendicular to the ray, $\vec{\nabla}_\perp n$, such that

$$\dot{\vec{e}}_{\dot{x}} = \frac{1}{n} \vec{\nabla}_\perp n = \vec{\nabla}_\perp \ln n.$$

Plug in n from part (c) and use $\ln(1-a) \approx -a$. What do you get for $\dot{\vec{e}}_{\dot{x}}$?

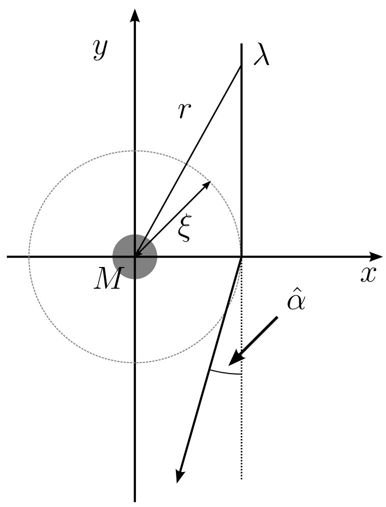
(h) The deflection angle $\hat{\alpha}$ is given by

$$\hat{\alpha} = \int \mathrm{d}\lambda \, (-\dot{\vec{e}}_{\dot{x}}).$$

What do you get if you use your knowledge of $\dot{\vec{e}}_{\dot{x}}$ from (g)?

(i) Your resulting integral can be cumbersome, since we would have to continuously integrate over the light path. However, since we assume the deflection angles to small, we can use the Born approximation from scattering theory, introducing the impact paramter ξ, and integrating over the unperturbed path:

$$\hat{\alpha}(\xi) = \frac{2}{c^2} \int_{-\infty}^{+\infty} \mathrm{d}\lambda \, \vec{\nabla}_\perp \Phi.$$



For simplicity, let's assume that ξ lies in the x–y–plane. Then the distance to the point–like mass at the origin is $r^2 = \xi^2 + \lambda^2$ (because λ is evaluated at the unperturbed path).
Taking $\Phi = -GM/r$, we get

$$\vec{\nabla}_\perp \Phi = \frac{GM}{r^3}\xi,$$

and thus for the deflection angle,

$$\hat{\alpha}(\xi) = \frac{2GM}{c^2}\xi \int_{-\infty}^{+\infty} \mathrm{d}\lambda \, \frac{1}{r^3}.$$

Evaluate the integral. Congratulations, your result for $\hat{\alpha}(\xi)$ is now the full general relativistic gravitational lensing deflection angle.

5. (5 points) **Extra: Sym-Metric**
   Why does the metric tensor $g_{\mu\nu}$ necessarily have to be symmetric? What implications would asymmetry have on the equations of motion?

"Astrophysics, you'll never be my closest friend // I find no comfort in what my mind can't comprehend"
– The Wombats, Tokyo (Vampires & Wolves)