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**Heidelberg University**
Björn Malte Schäfer

**General Relativity (MKTP3) Summer Term 2015**

**Exercise sheet 5** **12 May 2015**

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1. (15 points) **Curvature of a sphere**
   Last week, we calculated the Christoffel symbols on a sphere with fixed radius $r$. Just to remind you, the metric was

   $$g_{\mu\nu} = \mathrm{diag}(1, r^2, r^2 \sin^2(\theta)).$$

   (a) Calculate all non-vanishing entries of the Riemann tensor $R^{\mu}{}_{\nu\alpha\beta}$.
   (b) Calculate the Ricci-tensor $R_{\mu\nu}$.
   (c) Calculate the Ricci scalar $R$.

2. (10 points) **Energy–Momentum–Tensor**
   Given the Lagrangian

   $$L = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi),$$

   which is the Lagrangian of a field that obeys Klein–Gordon–equation, we're going to try and find out what the Energy–Momentum–tensor of a scalar field looks like.

   (a) Calculate $T_{\mu\nu}$ with

   $$T^{\mu\nu} = -2 \frac{\delta L}{\delta g^{\mu\nu}} - L g^{\mu\nu}.$$

   Tip: $\delta g^{\mu\rho} = -g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta}$.

   (b) Assume

   $$g_{\mu\nu} = \mathrm{diag}(c^2, -a^2(t), -a^2(t), -a^2(t))$$

   and that $\phi$ is spatially constant ($\partial_i \phi = 0$).
   Identify the density $\rho$ and the pressure $p$ of our field using

   $$T^{\mu}_{\nu} = \mathrm{diag}(\rho, -p, -p, -p).$$

   What assumptions are made with this diagonal form of $T^{\mu}{}_{\nu}$?

Well done, you just calculated the energy and pressure of a field that can drive cosmic inflation in the very early Universe!

3. (15 points) **An interesting line element II**
On the last sheet, you were asked to tediously calculate the Christoffel symbols of a line element $ds$ with

$$ds^2 = c^2dt^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2\right)\right].$$

Let's revisit this and see what we can further find out about this metric using the tools that we learnd in this week's lectures.

   (a) ~~Calculate the Riemann tensor!~~
   Just kidding, assume $k = 0$, and calculate the diagonal entries of the Ricci tensor $R_{\mu\mu}$. Remember:

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\beta_{\mu\nu}\Gamma^\alpha_{\beta\alpha} - \Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\beta\nu}$$

   (b) Find the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$.

   (c) Assuming once more that $T^\mu_{\ \nu} = \mathrm{diag}(\rho, -p, -p, -p)$, can you write down the 00-Einstein-field-equation, i.e.

$$G_{00} + g_{00}\Lambda = \frac{8\pi G}{c^4}T_{00}?$$

4. (5 points) **Extra: Spatial curvature of the Universe**
How would you go about designing an experiment orobservation that tests the flatness of the Universe as a whole? Can you distinguish between a flat and a non-flat cosmos?

*"Research starting from general notions, like the investigation we have just made, can only be useful in preventing this work from being hampered by too narrow views, and progress in knowledge of the interdependence of things from being checked by traditional prejudices.*
*This leads us into the domain of another science, of physics, into which the object of this work does not allow us to go to-day."*
***Bernhard Riemann's closing statement of his 'Habilitationsschrift' about the basic hypotheses of geometry, 1853, more than 50 years before the theory general relativity***