Heidelberg University
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General Relativity (MKTP3) Summer Term 2015

Exercise sheet 6 **→20 May← 2015**

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1. (15 points) **Density evolution**
Starting from a FLRW-metric, if we assume the universe to be filled with a perfect fluid, then the Einstein field equations can be reduced to just two equations, one of which you were asked to derive last week.
The two equations are called the *Friedmann equations*,

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3} - \frac{kc^2}{a^2}, \tag{1}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3\frac{p}{c^2}\right) + \frac{\Lambda c^2}{3}. \tag{2}$$

Here, a is the dimensionless scale factor, with today's value of $a_0 = 1$. The scale factor only depends on time and is what mathematically describes the apparent expasion of the universe. The energy-density of the universe, ρ, is coupled to it, meaning that it is influenced by the expansion, whilst in turn also *influencing the expansion.* This shows you the intrinsic non–linearity of general relativity.

   (a) What two symmetry assumptions are going in to the FLRW-metric? Are they sensible?

   (b) Show that it is possible to "absorb" The Λ-term into the energy-density of the universe ρ and the pressure p, and rewrite the Friedmann equations just in terms of k, ρ and w with $w = \frac{p}{\rho c^2}$.

   (c) Find a continuity equation! ($\dot{\rho} = \cdots$)

   (d) Show that for $w = const.$, integrating the continuity equation will get you a density-scale-relation
$$\rho = \rho_0 a^{-3(1+w)}. \tag{3}$$

   (e) For "ordinary matter" (i.e. galaxies, people, toasters,...), we have $w = 0$, for photons, we have $w = \frac{1}{3}$. Does the scale-evolution of ρ for the given w surprise you?

2. (15 points) **Density contributions**
We have seen in the previous exercise that it is (mathematically) possible to have the Λ-freedom absorbed in the density. This is essentially the same as arguing whether

it should be on the left-hand side or on the right-hand side of the Einstein field equations.
As it turns out, we can split off the density in all kinds of contributions, namely

$$\rho = \sum \rho_i = \rho_{\mathrm{matter}} + \rho_{\mathrm{photons}} + \rho_\Lambda + \rho_{\mathrm{neutrinos}} + \cdots$$

(a) Cosmologists call the ρ_{crit} the critical density, which is given by equation 1 for ρ if $k = 0$. Find ρ_{crit} in terms of H and G! Calculate today's critical density if $H_0 = 68\frac{\mathrm{km}}{\mathrm{s\,Mpc}}$.
Hint: $1\,\mathrm{Mpc} = 1000\,\mathrm{parsec} = 3.09 \times 10^{19}\,\mathrm{m}$.

(b) Now we define the relative matter contribution Ω,

$$\Omega := \frac{\rho}{\rho_{\mathrm{crit}}} = \sum \Omega_i = \Omega_{\mathrm{m}} + \Omega_{\mathrm{photons}} + \Omega_\Lambda + \Omega_{\mathrm{neutrinos}} + \cdots .$$

What does it mean for the Universe if

- $\Omega > 1$,
- $\Omega = 1$,
- $\Omega < 1$,

and the Friedmann equation (eq. 1) still needs to hold?

(c) Using the energy density contribution today $\Omega_0 = \frac{\rho_0}{\rho_{\mathrm{crit}}}$, show that you can re-write equation 1 using (equation 3) as

$$H^2(a) = H_0^2\left(\sum \Omega_{0,i} a^{-3(w+1)}\right) \equiv H_0^2 E^2(a). \tag{4}$$

(d) Assuming a flat universe made out of Ω_{m} and Ω_Λ, we have

$$\Omega_\Lambda + \Omega_{\mathrm{m}} = 1.$$

Re-write equaiton 4 as

$$\mathrm{d}t = \frac{\mathrm{d}a}{aH_0}\left(\sqrt{\Omega_{0,\Lambda} + \frac{1 - \Omega_{0,\Lambda}}{a^3}}\right)^{-1},$$

using $w_{\mathrm{m}} = 0$ and $w_\Lambda = -1$, and integrate from $a = 0$ to $a = a_0 = 1$ in order to find out the age of the Universe in the following cases:

- $\Omega_{0,\Lambda} = 0.7, \quad \Omega_{0,\mathrm{m}} = 0.3$,
- $\Omega_{0,\Lambda} = 0, \quad \Omega_{0,\mathrm{m}} = 1$.

Given that the oldest globular clusters are about 13 Gyr old, what does that say about the two toy models?

3. (10 points) **Distance measurements**

We have seen that measuring distance is general relativity can be an iffy task. In FLRW cosmology, there are a lot of different distances floating around in the literature. This exercise is supposed to give you a short summary of the most important ones.

We define the (radial) comoving distance to be

$$\chi(a) = c\int_a^1 \frac{\mathrm{d}a'}{a'^2 H(a')}. \tag{5}$$

(a) Assume $\Omega = \Omega_{\text{matter}} = 1$. What does $H(a') = H_0 E(a')$ look like now? Solve the integral analytically, and rename $\frac{c}{H_0} = \chi_{\mathrm{H}}$.

(b) The transverse comoving distance d_{M} is defined as

$$d_{\mathrm{M}} = \begin{cases} \frac{\chi_{\mathrm{H}}}{\sqrt{\Omega_{\mathrm{k}}}} \sinh\left(\sqrt{\Omega_{\mathrm{k}}}\frac{\chi(a)}{\chi_{\mathrm{H}}}\right) & \text{if } k > 0, \\ \chi(a) & \text{if } k = 0, \\ \frac{\chi_{\mathrm{H}}}{\sqrt{-\Omega_{\mathrm{k}}}} \sin\left(\sqrt{-\Omega_{\mathrm{k}}}\frac{\chi(a)}{\chi_{\mathrm{H}}}\right) & \text{if } k < 0. \end{cases}$$

Why is the transverse comoving distance different from the radial comoving distance if $k \neq 0$?

(c) The luminosity distance d_L is defined via the Flux F we measure from an object of luminosity L,

$$F = \frac{L}{4\pi d_L^2}.$$

If we measure the distance to the object in comoving distance, we can re-write this as

$$F = \frac{L(\chi)}{4\pi\chi^2(a)},$$

where L has to depend on χ. It turns out that

$$F = \frac{La^2}{4\pi\chi^2(a)},$$

and therefore

$$d_L = \chi(a)/a.$$

Give an argument, why $L(\chi) = La^2$!

(d) The angular diameter distance d_{A} of a structure that appears under the angle α and has a spatial extension l_0 is defined as

$$d_{\mathrm{A}} = \frac{l(a)}{\alpha},$$

where l_0 is also stretched along with the cosmic expansion, such that the comoving size is $l(a) = l_0/a$, i.e. $\alpha = \frac{l_0}{a\chi(a)}$, and thus

$$d_{\mathrm{A}} = a\chi.$$

Argue, why $l(a) = l/a$!

(e) There is a Python script on moodle called `distances.py` that will plot χ, d_L, and d_A against a.
Usage is simple enough: just type `> python distances.py` in your console, and it will produce a `distances.pdf` in the same directory.
This script will run with virtually all versions of python, you need to have some very standard packages installed (numpy, scipy, matplotlib). It runs perfectly well on the department's CIP-pools.

- Unfortunately, the inept programmer made an error while defining one the functions. Spot the error and correct it!
- Produce plots for three different cosmologies (i.e. sets of Ω_i) of your choosing.
- What happens if you change H_0?
- What is the remarkable feature about d_A?
- Extra: Set the y-axis to units of χ_H.
- Extra: Re-write the script so that it plots the distances against redshift z instead of scale factor a.

4. (5 points) **Extra: Redshift galore**
We've seen that there is a cosmological redshift $z = a^{-1} - 1$. If you were given the redshift value of a particular galaxy, say $z = 0.02$, how could you distinguish between the galaxy being redshifted due to its own, peculiar motion (relativistic doppler effect) z_pec and its "flow" in cosmic expansion z_c?

*"In the beginning the Universe was created.
This has made a lot of people very angry and been widely regarded as a bad move."*
Douglas Adams, The Restaurant at the End of the Universe