Heidelberg University
Björn Malte Schäfer

1	2	3	4	Σ

General Relativity (MKTP3) Summer Term 2015

Exercise sheet 7 $\rightarrow$**27 May**$\leftarrow$ **2015**

Tim Tugendhat (tugendhat@uni-heidelberg.de) Due: 9:15h, $\rightarrow$**1 June**$\leftarrow$ **2015**

Disclaimer: This sheet is fairly short as you don't have a full week to solve it.

1. (15 points) **Equation of state parameter meets scalar field**
   We're back on the Friedmann equations, which arose form the FLRW-metric.
   Now were going to delve into something called *Dark Energy*, which is essentially a generalised cosmological constant Λ. Last week we derived a continuity equation for the energy density of the cosmos,

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho(1+w) = 0, \tag{1}$$

   with the equation of state parameter $w = p/\rho c^2$.
   For $w = const.$, it lead us to an equation that describes the time evolution of the density depending on its particular w:

$$\rho = \rho_0 a^{-3(1+w)}.$$

   For Λ, we had $w = -1$, which meant $\rho_\Lambda = const.$, tautologically naming the cosmological constant.
   People have suggested that the cosmological constant – or whatever drives the acceleration of cosmic expansion – does not need to be constant in time, postulating $w = w(a)$.

   (a) Show that

$$\rho = \rho_0 \exp\left(-3\int_1^a \mathrm{d}(\ln a')[1 + w(a')]\right)$$

   solves equation 1 for a scale factor-dependent $w(a)$.

   (b) We have calculated that for a scalar field ϕ with a Lagrangian

$$L = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi),$$

   the pressure and energy density can – under the assumption of homogeneity and isotropy of the field – be written as

$$p = \frac{\dot{\phi}^2}{2} - V(\phi), \quad \rho c^2 = \frac{\dot{\phi}^2}{2} + V(\phi). \tag{2}$$

Therefore, the equation of state parameter w becomes

$$w = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)},$$

which, for a vanishing "kinetic" term $\dot{\phi} = 0$, mimics a cosmological constant, $w = -1$.
By plugging in the densities and pressures of a scalar field into equation 1, can you arrive at a equation of motion for ϕ? Does it remind you of the Klein-Gordon-equation for scalar fields at all?

2. (15 points) **Field equations**
Show that the Einstein field equations,

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4}T_{\mu\nu}, \tag{3}$$

could equally well be written as

$$R_{\mu\nu} = -\frac{8\pi G}{c^4}\left[T_{\mu\nu} - \frac{T}{2}g_{\mu\nu}\right] + \Lambda g_{\mu\nu}. \tag{4}$$

This shows the fundamentality of the "=" between G and T; you can not distinguish on what side of the equation your effect lies, meaning that a modified form of matter can always be described by a modified form of gravity.

3. (10 points) **A Hubble diagram for the 21th century: Supernovae Ia**
In the 1920s, Edwin Hubble noticed that there was a correlation of a galaxy's (or back then spiral nebulae) distance to us D and its apparent redshift $z = \frac{v}{c}$. His (empirical) equation

$$zc = H_0 D$$

related the apparent recession velocity to its distance, and Hubble concluded that distant objects were moving away from us.
Today, we know that it is not due to the inherent unattractiveness of our position in the Universe, but rather because of cosmic expansion.
Ever since Hubble, people wanted to extend his diagram to higher redshifts, but the problem was not the redshift measurement (which is simple enough). Determining the distance to a far-away object whose redshift is known is much more of a challenge.
For decades, people were looking for so-called *standard candles*: objects, that always have the same absolute brightness. With the knowledge of an object's absolute brightness, you can easily determine its (luminosity) distance by relating measured flux F and luminosity L,

$$F = \frac{L}{4\pi D^2}.$$

In the last 30 years, it has become apparent that Supernovae of type Ia are at least *"standardisable" candles*, as they always start burning as soon as their host star reaches a certain mass.[1] Therefore, Astronomers can deliver quasi-standard candles to cosmologists; the first couple of data points came in during the late 90s, and since then the sample has grown steadily. For the implications that we will see, i.e. a cosmic expansion that accelerates, the Nobel Prize in physics was awarded in 2011.

(a) As we have seen, we cannot distinguish between cosmic redshift and Doppler-redshift. How do we know that most of the objects actually follow cosmic flow and aren't just moving away?

(b) You can find a zip file on the moodle platform, `SNeIa.zip`. It contains `hubble.txt`, a text file containing data from the Union2 sample, a SNIa survey[2] and `SNeIa.py`, a python script that will plot a Hubble diagram for the sample and the luminosity distance that we defined on the last sheet.
Once again, using all of this is simple: just type `> python SNeIa.py` in your console, and it will produce a `SNeIa.pdf` in the same directory. Note that `hubble.txt` needs to also be present in the directory. This script will run with virtually all versions of python, you need to have some very standard packages installed (numpy, scipy, matplotlib). It runs perfectly well on the department's CIP-pools.

- What do you see in case of $\Omega_\mathrm{m} = 1$? Does it surprise you that people have not noticed this obvious clash sooner (1920s-30s)?
- How does changing h ($H_0 = h \cdot 100\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$) by $\sim 50\,\%$ affect the model? Can you explain the discrepancy between data and curve by changing it?
- Try to tweak the cosmological parameters such that the curve fits the data points better. Find (at least) two physically different models.
- How can you reconcile this degeneracy in parameter space? Is there a way you can determine the parameters independently without any other measurements?
- Extra: In this light, how do you interpret the diagram found at `http://supernova.lbl.gov/Union/figures/Union2.1_Om-Ol_slide.pdf`?
- Extra: For the python pros with too much time on their hands or just people who are interested: Try to implement a fitting algorithm in the script that determines the best fit for the constraint $\Omega_k = 0$.

4. (5 points) **Extra: CPT-symmetry**
Are the Einstein field equations CPT-invariant? I. e. do the physical processes stay the same if one were to transform $x^\mu \to -x^\mu$ and $q \to -q$?

"Cosmology brings us face to face with the deepest mysteries, questions that were once treated only in religion and myth." –***Carl Sagan***

[1] The Chandrasekhar mass, which is determined by electron degeneracy pressure, a fundamental physical effect that is the same throughout the Universe.

[2] credit: `http://supernova.lbl.gov/Union/`