Heidelberg University
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General Relativity (MKTP3) Summer Term 2015

Exercise sheet 9 9 June 2015

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1. (10 points) **Coordinate change**
   You're given the line element

$$\mathrm{d}s^2 = A(r)c^2\mathrm{d}t^2 - B(r)\mathrm{d}r^2 - C(r)r^2\mathrm{d}\Omega^2.$$

   Now put $B(r) = G(r) + C(r)$. Change the coordinates according to

$$\frac{\mathrm{d}\rho}{\rho} = \frac{\mathrm{d}r}{r}\sqrt{1 + \frac{G(r)}{C(r)}},$$

   where we introduce a new radial coordinate ρ, and show that the line element can be brought to the form of

$$\mathrm{d}s^2 = X(\rho)c^2\mathrm{d}t^2 - Y(\rho)\left(\mathrm{d}\rho^2 + \rho^2\mathrm{d}\Omega^2\right).$$

   (Tip: $X(\rho) = A(r)$, $Y(\rho) = \frac{r^2}{\rho^2}C(r)$.)
   We thus have recovered an spatially isotropic form of the metric!

2. (15 points) **Black hole evasion**
   A space ship is travelling at $\dot{r}_\infty = \frac{1}{\sqrt{2}}c$ (freely falling), when the crew notice that they are going to pass withing impact parameter $b = 4R_\mathrm{s}$ next to a black hole. (Assume their journey starts "infinitely away")
   The equation of motion was, as discussed on the last sheet,

$$\frac{1}{2}\dot{r}^2 + V_\mathrm{eff}(r) = const.,$$

   with the effective potential

$$V_\mathrm{eff}(r) = -\frac{GM}{r} + \frac{l^2}{2r^2} - \frac{GMl^2}{c^2r^3}.$$

   The angular momentum l is given by $v_\infty b$.
   Are they going to be captured by it or will they escape its pull?

3. (15 points) **Gravitational field of a rotating sphere**
   We have seen that the linearised field equations can be written as

$$\partial^\sigma\partial_\sigma h_{\mu\nu} = -\frac{16\pi G}{c^4}\left(T_{\mu\nu} - \frac{T}{2}\eta_{\mu\nu}\right).$$

The solution for h is an old friend from electrodynamics,

$$h_{\mu\nu}(\vec{r},t) = -\frac{4G}{c^4}\int \mathrm{d}^3r' \frac{U_{\mu\nu}(\vec{r}', t - |\vec{r}-\vec{r}'|/c)}{|\vec{r}-\vec{r}'|},$$

with $U_{\mu\nu} = T_{\mu\nu} - \frac{T}{2}\eta_{\mu\nu}$.
The energy-momentum-tensor can be written as

$$T^{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) u^\mu u^\nu - P\eta^{\mu\nu}.$$

Assume a sphere (radius R) with homogeneous density ρ. It's rotating with angular frequency ω. Assume $P \approx 0$, and only keep terms linear in $v/c = \omega R/c$.

Calculate the static fields $h_{\mu\nu}(\vec{r})$.

Tip: For static fields, we get (justify this!)

$$h_{\mu\nu}(\vec{r}) = -\frac{4G}{c^4}\int \mathrm{d}^3r' \frac{U_{\mu\nu}(\vec{r}')}{|\vec{r}-\vec{r}'|}.$$

4. (5 points) **Extra: Falling inside a black hole**
How long would a fall into a black hole take for a freely falling observer? What does his more fortunate friend at 'infinite' distance measure?

"Maybe we've spent too long trying to figure all this out with theory."
–***Amelia Brand in Interstellar***