Heidelberg University
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General Relativity (MKTP3) Summer Term 2015

Exercise sheet 10 17 June 2015
Tim Tugendhat (tugendhat@uni-heidelberg.de) Due: 9:15h, 22 June 2015
Disclaimer: Since this week, there hasn't been much new in the lecture, this sheet is shorter than usual. An extra and totally voluntary (numerical) exercise is coming up during the weekend, for those who are interested.[1]

1. (20 points) **Quadrupole moment of a rotating bar**
In the lecture we discussed that gravitational waves are generated by (at least) a changing *quadrupole* in the mass distribution. The quadrupole Q_{ij} is defined by

$$Q^{ij} = \int \mathrm{d}^3r\, x^i x^j \rho(\vec{r}).$$

(a) We're going to consider a solid body rotation by a 'dumbbell' around the x_3-axis. Let's consider a coordinate system in which this dumbbell is resting, and we're aligning our axes such that the quadrupole tensor Θ'^{ij} is diagonal,

$$\Theta'^{ij} = \int \mathrm{d}^3 r x'^i x'^j \rho'(\vec{r'}) = \mathrm{diag}(J_1, J_2, J_3).$$

Find a linear transformation $\alpha(t)$, such that

$$x^k = \alpha^k_l x'^l,$$

where the unprimed system x^k is resting in space, and that $z = z'$.
Tip: Write down $\alpha(t)$ as a 3×3 matrix containing $\sin(\tilde{\omega}t)$ etc.

(b) Show that

$$\Theta(t) = \alpha(t)\Theta'\alpha(t)^{\mathrm{T}}.$$

(c) Explicitly write down the components of $\Theta(t)$.

(d) Show that they are all of the form

$$\Theta^{ij} = const. + \left(Q^{ij} \exp(-2i\omega t) + \mathrm{compl.\,conj.}\right),$$

with

$$Q^{ij} = \frac{J_1 - J_2}{4}\left[\mathrm{i}\sigma_1 + \sigma_3\right].$$

σ are the Pauli matrices.

[1] That's also why this sheet is so late, setting up the numerical exercise is taking way longer to prepare than I had expected.

(e) We acknowledge that the only parts of Θ^{ij} that will generate gravitational waves are the temporally changing ones, of which Q^{ij} are the amplitude. Re-write the prefactor of Q^{ij} in terms of the moment of inertia with respect to the rotational axis J and the ellipticity ϵ,

$$J = J_1 + J_2, \quad \epsilon = \frac{J_1 - J_2}{J_1 + J_2}.$$

(f) Argue why $\omega = 2\tilde{\omega}$ is the 'real' angular frequency (Tip: What 'spin' does an ellpise have? Or: What 'spin' does a gravitational wave have?)

(g) The power radiated away by gravitational waves is given by

$$P = \frac{2G\omega^6}{5c^5}\left[\sum_{i,j=1}^{3}\left|Q^{ij}\right|^2 - \frac{1}{3}\left|\mathrm{tr}(Q)\right|^2\right].$$

Calculate P for our dumbbell in terms of ω, ϵ, and J.

2. (20 points) **And now for something completely different**
The power radiated away by gravitational waves of a stiff dumbbell is

$$P = \frac{32G\omega^6}{5c^5}\epsilon^2 J^2.$$

Let's look at a system of two masses M_1 and M_2 seperated by r. They are orbiting each other on a spherical orbit, such that $r = const.$, and

$$J = J_1 = \mu r^2, \quad J_2 = 0, \quad \epsilon = 1,$$

where μ is the reduced mass.

(a) Determine the angular frequency ω by requiring that the centrifugal force cancels the gravitational force.

(b) Write down P in terms of M_i and r!

(c) What is the total energy E of the system? (Tip: virial theorem)

(d) What is the energy loss per orbit in terms of v and c?

(e) Extra: Since $P = -\frac{\mathrm{d}E}{\mathrm{d}t}$, write down the differential equation you get for r!

(f) Extra: Substitute $x = (r/r(0))^4$, and set $\frac{\mathrm{d}x}{\mathrm{d}t} \equiv -\tau^{-1}$.

(g) Extra: What is the solution for x if you require that $x(0) = 1$? Retrieve $r(t)$.

3. (5 points) **Extra: Gravitational wave detectors**
Why is it so hard to build gravitational wave detectors?
Why have there not been any observations from gravitational waves?

"Oh, I'll just write them an intuitive and easy-to-use programme to simulate relativistic orbits in 2D, how long could that possibly take?"
*–**Me on June 12***