Heidelberg University
Björn Malte Schäfer

General Relativity (MKTP3) Summer Term 2015

Practice sheet 1 **17 April 2015**

Tim Tugendhat (tugendhat@uni-heidelberg.de) Not required to hand in!

1. (0 points) **Noether's theorem**
   Given the Lagrangian
   $$\mathcal{L}(r, \varphi, \dot{r}, \dot{\varphi}, t) = \frac{m}{2} \begin{pmatrix} \dot{r} & \dot{\varphi} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \begin{pmatrix} \dot{r} \\ \dot{\varphi} \end{pmatrix} - \frac{\alpha}{r}$$
   (a) Find $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}$. What does this mean?
   (b) Find $\frac{\mathrm{d}}{\mathrm{d}\varphi}\mathcal{L}$. What does this mean?

2. (0 points) **Virial theorem**
   A function $f(x)$ is said to be homogeneous of degree k iff $f(\alpha x) = \alpha^k f(x)$. Euler's homogeneous function theorem states that for a homogeneous function $f(\vec{x})$:
   $$\vec{x} \cdot \vec{\nabla} f(x) = k f(x).$$
   It is easy to see that the function
   $$T(\vec{v}) = \frac{m}{2} v^2$$
   is homogeneous of degree 2. Therefore, by the aforementioned theorem,
   $$\frac{\partial T}{\partial v_i} v_i := p_i v_i = 2T(\vec{v}).$$
   Here, we're using the Einstein summation convention and are implying summation over i and renamed the partial derivative p_i (momentum in i-direction). Let's rewrite the left-hand term as
   $$p_i v_i = \frac{\mathrm{d}}{\mathrm{d}t} (p_i x_i) - \dot{p}_i x_i$$
   If we average over time,
   $$\langle f \rangle = \lim_{t \to +\infty} \frac{1}{t} \int_0^t \mathrm{d}t' f(t'),$$

we get

$$2\langle T\rangle = \left\langle \frac{\mathrm{d}}{\mathrm{d}t}\left(p_i x_i\right) - \dot{p}_i x_i \right\rangle = \langle -\dot{p}_i x_i \rangle.$$

(a) Why does the first term vanish?

(b) Rewrite the remaining term as derivative of the potential $V(x)$.

(c) Apply Euler's theorem to the potential, as it is a homogeneous function of degree k as well.

(d) What does this imply for $\langle T\rangle$ and $\langle V\rangle$ in gravitational/electrostatic potentials ($V \propto r^{-1}$) and harmonic potentials ($V \propto r^2$)?

3. (0 points) **Gravitational "Opticks" – Let's lens like it's 1699**
"Do not Bodies act upon Light at a distance, and by their action bend its rays; and is not this action strongest at the least distance?"
– ***Isaac Newton, Opticks***

Gravitational lensing is a powerful tool in modern cosmology. With it, the matter contribution of the total matter-energy-content of the Universe can be measured to be about 30 %, the rest being the elusive Dark Energy. Lensing is sensitive to both "normal" matter and Dark Matter, unlike most other techniques.
In general relativity, light rays follow null geodesics, meaning that their path, too, can be 'bent' the presence of mass. The deflection angle $\hat{\alpha}$ in general relativity is given by

$$\hat{\alpha} = \frac{4GM}{c^2\xi}, \tag{1}$$

where ξ is the impact parameter (expressing how far away from the centre of mass they are passing).

Let us try and recreate this effect with Newtonian gravity:

(a)
- Calculate the escape velocity of a particle from a given potential $V = \frac{GmM}{r}$ from the simple argument that the initial kinetic energy T_i has to be the same as the initial potential energy V_i to escape.
- Set $v = c$ and solve the equation for r.

Congratulations, you've just calculated the Schwarzschild radius R_S, the event horizon for a static black hole. We will be needing it later.

(b) Let's consider the movement of a massive body (which will later be called light ray) in a Newtonian gravitational potential again,

$$\cancel{m}\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -G\frac{\cancel{m}M}{r^2}. \tag{2}$$

As we're interested in the body *reaching* us (i.e. not be *bound*) by the potential, we are interested in the solutions that are hyperbolæ. These shall be parametrised by

$$r = \frac{\xi(1+e)}{1+e\cos\varphi}, \tag{3}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\sqrt{GM\xi(1+e)}}{r^2} \tag{4}$$

where ξ is again the impact parameter (the nearest the trajectory comes to the centre of mass), and e is called eccentricity.

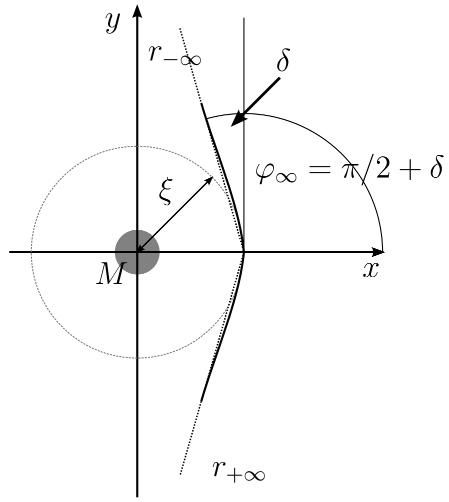

Figure 1: Here you can see the trajectory of our particle $\vec{r}$ in black, with the deflection angle $\hat{\alpha} = 2\delta$.

Writing the vector $\vec{r}$ in the following way,

$$\vec{r} = r\begin{pmatrix}\cos\varphi \\ \sin\varphi\end{pmatrix},$$

calculate the velocity $\vec{v} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}$ and its square, $|\vec{v}|^2$.

(c) Let's consider $r \to \infty$, i.e. when the object/light ray/signal reaches us. What condition can you then infer from equation 3 for the final angle φ_∞? If we define

$$\varphi_\infty = \frac{\pi}{2} + \delta,$$

what nifty relation do you get for $\sin\delta$ and e?

(d) Take your result from part (b) and set $v^2 = c^2$.
Now you can calculate the deflection angle $\hat{\alpha} = 2\delta$ for the Newtonian case. For this, we can assume small angles δ. Also, for example consider for the case of the sun the orders of magnitude in your fraction are,

- $c = 3 \times 10^8\,\mathrm{m\,s^{-1}}$,
- $G = 6.67 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-3}}$,
- $M_\odot = 2 \times 10^{30}\,\mathrm{kg}$, and
- $R_\odot = \xi = 7 \times 10^8\,\mathrm{m}$.

How does the fraction compare to unity? Can you simplify further? Are there any parallels to the general relativistic deflection angle? Calculate the Newtonian deflection angle of the sun at its surface!

4. (0 points) **Constant constants**
In the lecture, we've heard about the four "most" fundamental constants of modern physics, c, G, $\hbar$, and k_B.

(a) Which change would be most noticeable if one of those were suddenly to be multiplied by a factor of 2?

(b) Which of these would you consider "least" fundamental?