Statistical Physics - Exam Sheet

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1 Differential forms

- n-form α **exact** $\Leftrightarrow \alpha \in \mathrm{Im}(\mathrm{d}) \Leftrightarrow \exists\, n-1$-form β s.t. $\alpha = \mathrm{d}\beta$
  - β called **potential** for α; since $\mathrm{d}^2 = 0$, β not unique
- n-form α **closed** $\Leftrightarrow \alpha \in \ker(\mathrm{d}) \Leftrightarrow \mathrm{d}\alpha = 0$
- since $\mathrm{d}^2 = 0$, every exact form is closed; converse, i.e. if every closed form is exact, depends on topology: on contractible domain (such as $\mathbb{R}^n$), this holds by **Poincaré's lemma**
- some inexact differentials can be made exact by multiplying with **integrating factor** (and can then be integrated to give (path-independent) scalar field; useful in thermodynamics where T is i.f. that exactifies S)

2 Thermodynamics

- $\mathrm{d}S = \delta Q/T$ is path-independent total differential and state function (unlike path function δQ)
- ideal gas entropy: $S = S_0 + C_V \ln\big(\frac{T}{T_0}\big) + nR\ln\big(\frac{V}{V_0}\big)$
- **efficiency** $\eta = -\frac{W}{Q}$ defined as work *performed* by system (thus minus sign) divided by heat *added*
  - all reversible engines equally efficient
  - ideal heat engine pumping between reservoirs T_c, T_h: $\eta = 1 - \frac{T_c}{T_h} = 1 - \big(\frac{V_{\min}}{V_{\max}}\big)^{\frac{nR}{C_V}}$, where $\frac{nR}{C_V} = \frac{C_p - C_V}{C_V} = \gamma - 1$
  - isochoric heat change in ideal gas: $Q_{if} = C_V(T_f - T_i)$
  - by second law, reversible process is isentropic, $\oint \mathrm{d}S = 0$
- for **paramagnets** with first law $\mathrm{d}U = T\,\mathrm{d}S + H\,\mathrm{d}M$ obeying **Curie's law** $M = \frac{C}{T}H$, internal energy depends only on T
- **potentials**: internal energy $U = TS - pV$, Helmholtz free energy $F = U - TS = -pV$, $H = U + pV = TS$, Gibbs free energy $G = U + pV - TS$ (add $\sum_i \mu_i N_i$ everywhere in presence of chemical potentials)
  - differentials i.t.o. natural variables: $\mathrm{d}U = T\mathrm{d}S - p\mathrm{d}V$, $\mathrm{d}F = -S\mathrm{d}T - p\mathrm{d}V$, $\mathrm{d}H = T\mathrm{d}S + V\mathrm{d}p$, $\mathrm{d}G = -S\mathrm{d}T + V\mathrm{d}p$
  - all potentials related via Legendre transformation
  - **principle of minimum energy**: entropy maximizes in equilibrium, all potentials extremize: U minimizes for fixed S, V, (follows from first and second law); F minimizes at fixed T, V; H at fixed p; G at fixed T and p
- **Maxwell relations**: $U \to \frac{\partial T}{\partial V}\big|_S = -\frac{\partial p}{\partial S}\big|_V$, $F \to \frac{\partial S}{\partial V}\big|_T = \frac{\partial p}{\partial T}\big|_V$, $H \to \frac{\partial T}{\partial p}\big|_S = \frac{\partial V}{\partial S}\big|_p$, $G \to \frac{\partial S}{\partial p}\big|_T = -\frac{\partial V}{\partial T}\big|_p$
  - related via $\frac{\partial x}{\partial y}\big|_z = \big(\frac{\partial y}{\partial x}\big|_z\big)^{-1}$ and $\frac{\partial x}{\partial y}\big|_z \frac{\partial y}{\partial z}\big|_x \frac{\partial z}{\partial x}\big|_y = -1$
- **intensive quantities**: temperature T, pressure p, density ρ, chemical potential μ, concentration c, magnetic permeability μ_m, melting/boiling point, specific heat capacity c_V, specific volume v; **extensive**: particle number N, internal energy U, enthalpy H, entropy S, Gibbs energy G, heat capacity C_V, Helmholtz energy F, mass m, volume V
- **real gases**: **van der Waals** $p + \frac{a}{v^2} = \frac{RT}{v-b}$ or $p + \frac{n^2 a}{V^2} = \frac{nRT}{V-nb}$, with $n = \frac{N}{N_\mathrm{A}}$, $R = N_\mathrm{A} k_\mathrm{B}$; **Dieterici** $p = \frac{RT}{v-b}\exp\big(-\frac{a}{vRT}\big)$; **virial expansion** $\frac{pv}{RT} = 1 + \sum_{i=1}^\infty \frac{B_i}{v^i}$ for large volume v
- **useful identities**: $p = -\frac{\partial F}{\partial V}\big|_T$, $S = -\frac{\partial F}{\partial T}\big|_V$, $\frac{1}{T} = \frac{\partial S}{\partial U}$, $t\frac{\partial U}{\partial V}\big|_T = T\,\frac{\partial p}{\partial T}\big|_V - p$; **Helmholtz eq.**: $\frac{\partial U}{\partial V}\big|_T = T^2\,\frac{\partial}{\partial T}\,\frac{p}{T}\big|_V$; **thermal expansion coeff.** $\alpha = \frac{1}{V}\,\frac{\partial V}{\partial T}\big|_{a,N}$; (isothermal) **compressibility** $\kappa_T = -\frac{1}{V}\frac{\partial p}{\partial V} = \frac{V^2}{N^2}\frac{\partial^2 p}{\partial^2 \mu}$; **$n$-ball**: volume $V_n(R) = \frac{\pi^{n/2}R^n}{\Gamma(\frac{n}{2}+1)}$, area $A_{n-1}(R) = \partial_R V_n(R) = \frac{2\pi^{n/2}R^{n-1}}{\Gamma(\frac{n}{2})}$
- **equipartition thm.**: in equilibrium, all microstates compatible with macroscopic conditions equiprobable; energy shared equally amongst all d.o.f.s (i.e. equilibrated systems maximize entropy), each quadratic d.o.f. carries energy $\frac{k_\mathrm{B}T}{2}$
- **heat engine**: device that transfers heat from hot to cold reservoir $T_c < T_h$ with aim of converting as much of transferred heat into mechanical work as possible; maximum efficiency attained by *reversible* engine operating idealized Carnot cycle $\eta_\mathrm{C} = 1 - \frac{T_c}{T_h}$; **heat pump**: moves heat in other direction from cold to hot, goal now to use as little mechanical work as possible to do so, i.e. heat engine operating in reverse $\Rightarrow$ maximum efficiency of reversible heat pump given by reciprocal of η_C, $\eta_\mathrm{hp} = \frac{T_h}{T_h - T_c}$
- **black-body radiation**: caloric e.o.s. $U = u(T)V$, thermal e.o.s. $p = \frac{u(T)}{3}$, Stefan-Boltzmann law $u(T) = \sigma T^4$, Gibbs fundamental relation (first law) $\mathrm{d}U = T\mathrm{d}S - p\mathrm{d}V$ yields entropy $S(T,V) = S_0 + \frac{4}{3}\sigma V T^3$, adiabatic e.o.s. $\delta Q = T\mathrm{d}S = 0 \Rightarrow S \propto VT^3 = \text{const.}$

3 Statistics

ensemble	*fixed*	*distribution*	*interpretation*
micro	$V\ N\ E$	$\frac{1}{Z_m}\delta(H-E)$	isolated system
canonical	$V\ N\ T$	$\frac{1}{Z_c}e^{-\beta H}$	system + heat bath
grand	$V\ T\ \mu$	$\frac{1}{Z_g}e^{-\beta(H-\mu N)}$	$\uparrow$ + particle reservoir

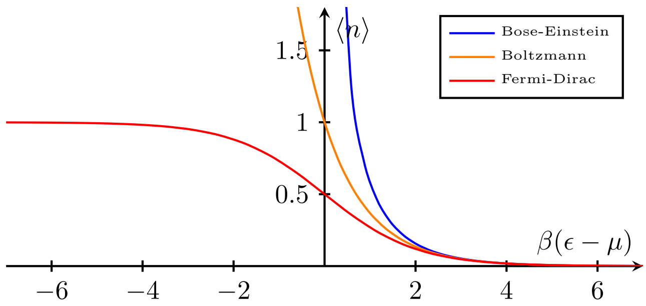
- entropies: **Boltzmann** $S_m = k_B \ln Z_m$, $S_c = k_B(\ln Z_m + \beta\langle H\rangle_c)$, $S_g = k_B[\ln Z_g + \beta\langle H\rangle_g + \beta\mu\langle N\rangle_g]$ (apply only to equilibrium), **Gibbs** $S_\mathrm{G} = -k_\mathrm{B}\sum_i p_i \ln(p_i)$ respectively $S_\mathrm{G} = -k_\mathrm{B}\int_\Gamma \omega \ln\omega \mathrm{d}\mu$ on discrete/continuous measure space (valid generally), S_G constant in time due to Liouville's eq. $\frac{\mathrm{d}\omega}{\mathrm{d}t} = 0$
  - **Liouville thm.** phase space distribution constant along any trajectory in Γ; viewing motion through phase space as incompressible 'fluid flow' of conserved current of system points implies via Noether's thm. invariance under time evolution generated by Noether charge Hamiltonian
- **equivalence of ensembles**: thermodynamics same for large class of systems (incl. ideal gas), $\frac{S_m}{N} \overset{N\to\infty}{\longleftrightarrow} \frac{S_c}{N}$, $\frac{S_m}{V} \overset{V\to\infty}{\longleftrightarrow} \frac{S_g}{V}$; warning: fails in important cases e.g. phase transitions

$$\begin{array}{ccccc} Z_m(E) & \xrightarrow{\text{Laplace in } E} & Z_c(\beta) & \xrightarrow{\text{Laplace in } N} & Z_g(\mu) \\ \downarrow & & \downarrow & & \downarrow \\ \sigma = \frac{S_m}{N} & \xrightarrow{\text{Legendre in } \epsilon = \frac{E}{N}} & f = \frac{F}{N} & \xrightarrow{\text{Legendre in } \rho = \frac{N}{V}} & \frac{\Omega}{V} \end{array}$$

- **Hamilton's eqs.** $\dot{q}_i = \frac{\partial H}{\partial p_i}$, $\dot{p}_i = -\frac{\partial H}{\partial q_i}$ where $p_i = \frac{\partial L}{\partial \dot{q}_i}$
- **Boltzmann eq.** $\big(\frac{\partial}{\partial t} + \frac{\boldsymbol{p}}{m}\cdot\boldsymbol{\nabla}_x + \boldsymbol{F}\cdot\boldsymbol{\nabla}_p\big)f(\boldsymbol{x},\boldsymbol{p},t) = \int \mathrm{d}^3k\,\mathrm{d}^3p'\mathrm{d}^3k'\quad |\langle \boldsymbol{p}',\boldsymbol{k}'|T|\boldsymbol{p},\boldsymbol{k}\rangle|^2\left[f_{p'}\,f_{k'} - f_p f_k\right]$ describes (non-equilibrium) evolution of phase space densities (for large systems, e.g. gases); incorporates **molecular chaos**, which assumes collision term results solely from two-body collisions between particles uncorrelated prior to scattering (breaks time-reversal invariance since they are obviously correlated after collision), m.c. allows writing collision term as $\boldsymbol{p}$-integral in which two-particle correlator $F(\boldsymbol{x},\boldsymbol{p},\boldsymbol{k},t)$ factorizes into one-particle distributions $f(\boldsymbol{x},\boldsymbol{p},t)\,f(\boldsymbol{x},\boldsymbol{k},t)$

- for stationary systems, Boltzmann eq. greatly simplifies to $\frac{\boldsymbol{p}}{m}\cdot\boldsymbol{\nabla}_x f_0(\boldsymbol{x},\boldsymbol{p}) = -\boldsymbol{F}(\boldsymbol{x})\cdot\boldsymbol{\nabla}_p f_0(\boldsymbol{x},\boldsymbol{p})$ since 1. distributions $f(\boldsymbol{x},\boldsymbol{p})$ lose explicit time-dependence 2. H-function becomes stationary, resulting in detailed balance, meaning number of particles leaving any mode due to given scattering process equals number entering that mode by reverse process; stationary solution: **Boltzmann distr.** $f_0(\boldsymbol{x},\boldsymbol{p}) \propto \exp\big[-\beta\big(\frac{\boldsymbol{p}^2}{2m}+V(\boldsymbol{x})\big)\big]$

- **Boltzmann H-fct.** $H(t) = \int \mathrm{d}^3x\,\mathrm{d}^3p\, f(\boldsymbol{x},\boldsymbol{p},t)\ln[f(\boldsymbol{x},\boldsymbol{p},t)]$
- **Poincaré recurrence thm.** in context of ergodicity (time avr. = phase space avr.), isolated phase-space volume preserving (e.g. Hamiltonian) systems will return arbitrarily close to initial state after finite time
- **Gibbs paradox**: for classical counting of states, entropy not extensive: S increases when gas of identical particles mixed with itself (fixable by adding $\frac{1}{h^{3N}N!}$ to the measure $\mathrm{d}\mu_m$); entropy from Boltzmann's H does not have this problem
- **Gibbs variational principle**: microcanonical state has maximal entropy among all states on energy shell $\mathcal{E}$, holds for other ensembles as well
  - useful inequality: $f(\ln f - \ln g) \geq f - g\ \forall\, f,g \geq 0$ (= for $f = g$), follows from $x\ln x \geq x - 1$ for $x \geq 0$
- **canonical part. fct.** discrete classical $Z_c = \sum_i e^{-\beta E_i}$, continuous classical $Z_c = \int_\Gamma e^{-\beta H}\,\mathrm{d}\mu_c$ with $\mathrm{d}\mu_c = \frac{\mathrm{d}^{3N}q\,\mathrm{d}^{3N}p}{h^{3N}N!}$; discrete quantum $Z = \mathrm{tr}_{\mathcal{H}}(e^{\beta\hat{H}})$
  - internal energy $U = -\partial_\beta \ln(Z_c)$, free energy $F = -\frac{1}{\beta}\ln(Z_c)$
- **grand part. fct.** $Z_c = \sum_{N=0}^\infty \int_\Gamma g_i e^{-\beta(H-\mu N)}\,\mathrm{d}\mu_c = \sum_{N=0}^\infty z^N Z_c(N)$, with fugacity $z = e^{\beta\mu}$ and e.e.v. $\langle H\rangle_g = \frac{1}{Z_g}\sum_{N=0}^\infty \int_\Gamma H\, e^{-\beta(H-\mu N)}\,\mathrm{d}\mu_c$, expected particle number $\langle N\rangle = \frac{1}{\beta}\frac{\partial}{\partial\mu}\ln Z_g = -\frac{\partial\Omega}{\partial\mu}$
- **fluctuations**: e.g. energy $\sigma_H^2 = \langle(H-\langle H\rangle)^2\rangle = \langle H^2\rangle - \langle H\rangle^2 = -\frac{\partial\langle H\rangle}{\partial\beta}$, where $\langle H\rangle = \frac{1}{Z_c}\int_\Gamma H\, e^{-\beta H}\,\mathrm{d}\mu_c = -\frac{1}{Z_c}\frac{\partial Z_c}{\partial\beta}$
- **uncoupled Ising model/ideal paramagnet** (N uncoupled Ising spins in external field h): $H(s) = -hm\sum_{j=1}^N s_j = -hM(s)$, minus $\Rightarrow$ energy lowered if spins align with external field; $s = (s_1,\ldots,s_N) \in \mathcal{S}_N = \{\pm1\}^N$ is any of 2^N possible spin configurations; energy shell $\mathcal{E} = \{s \in \mathcal{S}_N | H(s) = E\}$ (non-empty only if $E/hm \in \mathbb{Z}\cap[-N,N]$), microcan. part. fct. $Z_m = \binom{N}{N_\downarrow}$, entropy $S = Nk_\mathrm{B}\ln(2) - \frac{Nk_\mathrm{B}}{2}\big[\big(1+\frac{E}{Nmh}\big)\ln\big(1+\frac{E}{Nmh}\big) + \big(1-\frac{E}{Nmh}\big)\ln\big(1-\frac{E}{Nmh}\big)\big]$; canonical p.f. $Z_c = \sum_{s\in\mathcal{S}_N} e^{-\beta H(s)} = \prod_{j=1}^N\sum_{s_j} e^{\beta hms_j} = [2\cosh(\beta hm)]^N$, free energy $F = -\frac{1}{\beta}\ln Z_c = -\frac{N}{\beta}\ln[2\cosh(\beta hm)]$, magnetization $M = -\frac{\partial F}{\partial h} = Nm\tanh(\beta mh)$, caloric e.o.s. $E = -\frac{\partial\ln Z_c}{\partial\beta} = -Nmh\tanh(\beta mh)$, specific heat $c_h = \frac{C_h}{N} = \frac{1}{N}\frac{\partial U}{\partial T}\big|_h = k_\mathrm{B}\frac{\beta^2h^2m^2}{\cosh^2(\beta hm)}$, magnetic susceptibility $\chi = \frac{\partial M}{\partial h} = \frac{\beta Nm^2}{\cosh^2(\beta hm)}$, Gibbs fundamental rel. $\mathrm{d}S = \frac{1}{T}\mathrm{d}E + \frac{M}{T}\mathrm{d}h - \frac{\mu}{T}\mathrm{d}N$
- **coupled Ising model** $H(s) = J\sum_{\langle s_i,s_j\rangle}^N(1 - s_is_j) - h\sum_{i=1}^N s_i$, can. part. fct. $Z_c = \sum_{s\in\mathcal{S}_N} e^{-\beta H(s)}$, **mean-field magnetization** per spin of Ising ferromagnet $m = \tanh(2d\,\beta\,J\,m + \beta\,h)$ with crit. temp. $T_c = 2d\,J/k_\mathrm{B}$
- **density operator**: $\rho^\dagger = \rho$, $\mathrm{tr}(\rho) = 1$, $\rho \geq 0$, observable A expectation value $\langle A\rangle = \mathrm{tr}(\rho A)$
  - spin-$\frac{1}{2}$: $\rho = \frac{1}{2}\big(\mathbb{1}_2 + \boldsymbol{a}\cdot\boldsymbol{\sigma}\big)$ with Bloch vector $\boldsymbol{a} = \langle\boldsymbol{\sigma}\rangle = \mathrm{tr}(\boldsymbol{\sigma}\rho)$, eigenvalues $\lambda_\pm = \frac{1}{2}\big(1 \pm |\boldsymbol{a}|\big)$ where $|\boldsymbol{a}| \leq 1$; pure state, i.e. $\rho^2 = \rho$ if $|\boldsymbol{a}| = 1$, mixed else, spin polarization $\pi = \frac{p_+ - p_-}{p_+ + p_-}$
  - **von Neumann entropy** $S_\mathrm{N} = -k_\mathrm{B}\,\mathrm{tr}_{\mathcal{H}}[\rho\ln(\rho)]$ with $\rho = \sum_n p_n|n\rangle\langle n|$ sum over complete set of states of Hilbert $\mathcal{H}$, reduces to Boltzmann entropy for microcan. ensemble
  - canonical density op. $\rho_c = e^{\beta(F-H)} = \frac{1}{Z_c}e^{-\beta H}$, with normalization $Z_c = \mathrm{tr}\, e^{-\beta H}$
- (anti-)symmetrized **many-particle states** $\mathcal{P}_\pm|\alpha_1\rangle\otimes\cdots\otimes|\alpha_n\rangle = \frac{1}{n!}\sum_{\pi\in S_n}(\pm1)^\pi|\alpha_{\pi(1)}\rangle\otimes\cdots\otimes|\alpha_{\pi(n)}\rangle$ furnish Fock space $\mathcal{F}_\pm(\mathcal{H}) = \bigoplus_{n=0}^\infty \mathcal{H}_\pm^n$ with $\mathcal{H}_\pm^n = \mathcal{P}_\pm\,\mathcal{H}^{\otimes n}$
- **occupation numbers** $\langle n_k^\pm\rangle_g = -\frac{1}{\beta}\frac{\partial}{\partial\epsilon_k}\ln(Z_g^\pm) = \frac{1}{e^{\beta(\epsilon_k-\mu)}\mp1}$ given by Bose-Einstein/Fermi-Dirac distributions with $Z_g^\pm = \mathrm{tr}_{\mathcal{F}_\pm}\big(e^{-\beta(\hat{H}-\mu\hat{N})}\big) = \prod_{j=0}^\infty\sum_{n_j^\pm=0}^{\infty\text{ or }1} e^{-\beta\,n_j(\epsilon_j-\mu)} = \prod_{j=0}^\infty\big(1\mp e^{-\beta(\epsilon_j-\mu)}\big)^{\mp1}$
  - Maxwell-Boltzmann distribution $f(\boldsymbol{p}) = e^{-\beta\epsilon(\boldsymbol{p}-\mu)}$, or normalized and spherically sym. $f(v) = \big(\frac{m}{2\pi kT}\big)^{3/2}4\pi v^2 e^{-\frac{mv^2}{2k_\mathrm{B}T}}$



- total particle fluctuations $\Delta N = \langle N^2\rangle - \langle N\rangle^2 = \sum_k\langle n_k^2\rangle - \sum_k\langle n_k\rangle^2$ for non-interacting gases
- **grand potential** $\Omega = -\frac{1}{\beta}\ln Z_g$, $\Omega = -pV$, ideal Bose/Fermi gas $\Omega^\pm(T,V,\mu) = -\frac{1}{\beta}\ln(Z_g^\pm) = \pm\frac{1}{\beta}\sum_j\ln\big(1\mp e^{-\beta(\epsilon_j-\mu)}\big)$ with $p = -\frac{\partial\Omega}{\partial V}$, $S_g = -\frac{\partial\Omega}{\partial T}$
- **ideal Fermi gas**: pressure $\beta p = \frac{4\pi g_s}{h^3}\int_0^\infty \mathrm{d}p\,p^2\ln\big(1 + e^{\beta(p^2/2m-\mu)}\big) = \frac{g_s}{\lambda^3}f_{5/2}^-(z)$, density $\rho = \frac{N}{V} = \frac{4\pi g_s}{h^3}\int_0^\infty\frac{p^2\,\mathrm{d}p}{1+e^{\beta(p^2/2m-\mu)}} = \frac{g_s}{\lambda^3}f_{3/2}^-(z)$, where $f_\nu^\pm(z) = \frac{1}{\Gamma(\nu)}\int_0^\infty\frac{x^{\nu-1}\,\mathrm{d}x}{z^{-1}e^x\mp1}$, $z = e^{\beta\mu}$ and $\lambda = h/\sqrt{2\pi mk_\mathrm{B}T}$, low temperature/high density means $\rho\lambda^3 \gg 1 \Rightarrow z \gg 1$, e.o.s. $\frac{pV}{k_\mathrm{B}T} = \ln Z_g^\pm = \mp g_s\sum_k\ln\big(1\mp e^{-\beta(\epsilon_k-\mu)}\big)$, energy $U = \frac{3}{2}pV$
- **ideal Bose gas**: non-interacting (spinless) bosons with disp. rel. $\epsilon_k = \hbar^2k^2/2m$ contained in $V = L^d$; grand part. fct. $Z_g = \sum_{N=0}^\infty\sum_{\{n_k\}} e^{-\beta\sum_k n_k(\epsilon_k-\mu)}\delta_{\sum_k n_k,N} = \prod_k\sum_{\{n_k\}} e^{-\beta\sum_k n_k(\epsilon_k-\mu)} = \prod_k\frac{1}{1-e^{-\beta(\epsilon_k-\mu)}}$, grand potential $\Omega = -\frac{1}{\beta}\ln(Z_g) = \frac{1}{\beta}\sum_k\ln\big(1-e^{-\beta(\epsilon_k-\mu)}\big) \xrightarrow{V\to\infty} \frac{V}{\beta}\frac{A_{d-1}(1)}{(2\pi)^d}\int_0^\infty\ln(1 - ze^{-\beta\epsilon_k})k^{d-1}\mathrm{d}k$, now subst. $x = \frac{\beta\hbar^2k^2}{2m}$ followed by partial integration to get $\Omega = \frac{V}{\beta}\frac{A_{d-1}(1)}{(2\pi)^d}\frac{1}{2}\big(\frac{2m}{\hbar^2\beta}\big)^{\frac{d}{2}}\int_0^\infty x^{\frac{d}{2}-1}\ln(1 - ze^{-x})\mathrm{d}x = -\frac{V}{\beta}\frac{A_{d-1}(1)}{d}\big(\frac{2m}{h^2\beta}\big)^{\frac{d}{2}}\int_0^\infty\frac{x^{\frac{d}{2}}}{z^{-1}e^x-1}\mathrm{d}x = -\frac{V}{\beta\lambda^d}f_{\frac{d}{2}+1}^+(z)$, avr. number of particles $N = \frac{\partial\ln\Omega}{\partial(\beta\mu)} = \frac{V}{\lambda^d}f_{\frac{d}{2}}^+(z) \Rightarrow$ density $n(z) = \frac{1}{\lambda^d}f_{\frac{d}{2}}^+(z)$; pressure $pV = -\Omega$ and $E = -\frac{\partial\ln Z_g}{\partial\beta} = \frac{d}{2}\frac{\ln Z_g}{\beta} = -\frac{d}{2}\Omega$, i.e. $E = \frac{d}{2}pV$; critical temperature obtained by solving $n(1) = \frac{1}{\lambda^d}\zeta(\frac{d}{2})$ for T_c, for $d > 2$, this yields $T_c = \frac{h^2}{2mk_\mathrm{B}}\big(\frac{n}{\zeta(\frac{d}{2})}\big)^{\frac{2}{d}}$; heat capacity at $T < T_c$, $z = 1$ is $C(T) = \frac{\partial E}{\partial T}\big|_{z=1} = k_\mathrm{B}\frac{d}{2}(\frac{d}{2}+1)\frac{V}{\lambda^d}\zeta(\frac{d}{2}+1)$ e.o.s. at high T $(z\to0)$ is $\frac{pV}{Nk_\mathrm{B}T} = 1 - \frac{N\lambda^3}{4\sqrt{2}V} + \ldots$ has correction to ideal gas that lowers the pressure and is solely due to quantum statistics (not interactions)