General Relativity – Exam Sheet

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Formula summary by Janosh Riebesell

Newton's axioms: 1. free particles move along straight lines at constant velocity, 2. $\vec{F} = m\ddot{\vec{x}}$, 3. actio = −reactio

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Galilei transformations: universal time $t = t'$ ∀ observers; 2 frames S, S' in relative motion ∥ x-axis: $x_1' = x_1 - vt$, $x_2' = x_2$, $x_3' = x_3$,
$u_1' = \frac{\mathrm{d}x_1'}{\mathrm{d}t} = u_1 - v$; $a_1' = \frac{\mathrm{d}u_1'}{\mathrm{d}t} = a_1 \implies$ accelerations identical in all frames, particles free in S are free in S'

Lorentz transformations: $t \neq t'$, based on two assumptions: 1. spacetime homogeneous, 2. $c = c'$ ∀ observers. Classification of events valid in all frames: $\mathrm{d}s^2 > 0$ timelike, $\mathrm{d}s^2 = 0$ lightlike, $\mathrm{d}s^2 < 0$ spacelike

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Proper time τ displayed by comoving clock; $\mathrm{d}s^2 = c^2\,\mathrm{d}\tau^2$, $\mathrm{d}\tau = \frac{1}{\gamma}\,\mathrm{d}t$

(Weak) equivalence principle: $m_i = m_g$, i.e. inertial and gravitational mass of any object are equal

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Length of a curve C parametrized by λ: $I = \int_C \mathrm{d}s = \int_C \sqrt{|g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu|} = \int_a^b \mathrm{d}\lambda \sqrt{|g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}|}$, area: $\mathrm{d}A = \sqrt{|g_{\mu\nu}|}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \overset{2d}{=} \sqrt{|g_{11}g_{22}|}\,\mathrm{d}x^1\,\mathrm{d}x^2$

Riemannian manifolds are locally Cartesian, i.e. can locally be matched to Minkowski space

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Christoffel symbols (affine connection): $\Gamma^\rho_{\mu\nu} = \frac{1}{2}\,g^{\rho\sigma}\big(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\big)$, can be obtained by diff. $g_{\mu\nu} = \vec{e}_\mu \cdot \vec{e}_\nu$ w.r.t. x^ρ and cyclically perm. indices; $\Gamma^\rho_{\mu\nu} = \vec{e}^{\,\rho}\frac{\partial}{\partial x^\nu}\vec{e}_\mu$; equivalence: locally Cartesian ↔ vanishing connection; torsion $T^\alpha_{\mu\nu} \coloneqq \Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu}$

Cov. derivative of contravector $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho}V^\rho$, of covector $\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\rho_{\mu\nu}V_\rho$; $\nabla_\mu V^\nu = \partial_\mu V^\nu$ if locally Cartesian

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Geodesic (on torsion-free manifold): shortest connection of two points, more generally curve with constant tangent. Geodesic equation (eq. of parallel transport): $\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda} = 0 = \frac{\mathrm{d}u^\mu}{\mathrm{d}\lambda} + \Gamma^\mu_{\rho\sigma}u^\rho u^\sigma$, path-dependent due to varying $\Gamma^\mu_{\rho\sigma}$

Riemann tensor $R^\rho_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$, $R_{\rho\sigma\mu\nu} = g_{\rho\lambda}R^\lambda_{\sigma\mu\nu}$; symmetries: $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu} = R_{\sigma\rho\nu\mu}$, $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$, $R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0$, $d = \frac{n^2}{12}\,(n^2-1)$ indep. comp.; metric's comp. constant ⟺ $R^\rho_{\sigma\mu\nu} = 0$ ⟺ space flat

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Ricci tensor $R_{\mu\nu} = R^\lambda_{\mu\lambda\nu} = R_{\nu\mu}$, Ricci scalar $R = R^\mu_\mu = g^{\mu\nu}R_{\mu\nu}$

Bianchi identity $\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} = 0$; Riemann tensor on 2-sphere $R^i_{jkm} = \frac{1}{r^2}\left(\delta^i_k g_{jm} - \delta^i_m g_{jk}\right)$

Energy-mom. tensor of perfect fluid (:= isotropic in its rest frame) $T^{\mu\nu} = \big(\rho + \frac{p}{c^2}\big)u^\mu u^\nu - p\eta^{\mu\nu} = \mathrm{diag}(\rho c^2, p, p, p)$

Einstein field eqs. $G_{\mu\nu} = R_{\mu\nu} - \frac{R}{2}\,g_{\mu\nu} = \frac{8\pi G}{c^4}\,T_{\mu\nu}$

FLRW: $\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2\right]$, inv. under $k \to k/|k|$, $r \to \sqrt{|k|}r$, $a \to a/\sqrt{|k|}$; metric singular at $r = \pm 1$

- $k = -1 \leftrightarrow$ negative curvature ↔ open universe ↔ $\rho < \rho_{\mathrm{crit}} \leftrightarrow \Omega < 1$
- $k = 0 \leftrightarrow$ no curvature ↔ flat universe ↔ $\rho = \rho_{\mathrm{crit}} \leftrightarrow \Omega = 1$
- $k = 1 \leftrightarrow$ positive curvature ↔ closed universe ↔ $\rho > \rho_{\mathrm{crit}} \leftrightarrow \Omega > 1$

For $k = \pm 1 \to$ new radial coordinate $r = \sin\chi$ $(k = 1)$ or $r = \sinh\chi$ $(k = -1)$, $\frac{\mathrm{d}r}{\mathrm{d}\chi} = \sqrt{1 \mp r^2}$; $\rho_{\mathrm{crit}} = \frac{3H^2}{8\pi G}$;
$\Gamma^1_{11} = \frac{kr}{1-kr^2}$, $\Gamma^0_{11} = \frac{a\dot{a}}{c(1-kr^2)}$, $\Gamma^0_{22} = \frac{a\dot{a}r^2}{c}$, $\Gamma^0_{33} = \frac{a\dot{a}r^2}{c}\sin^2\theta$, $\Gamma^1_{01} = \Gamma^1_{10} = \Gamma^2_{02} = \Gamma^2_{20} = \Gamma^3_{30} = \Gamma^3_{03} = \frac{\dot{a}}{ca}$, $\Gamma^1_{22} = -r(1-kr^2)$, $\Gamma^1_{33} = \Gamma^1_{22}\sin^2\theta$, $\Gamma^2_{33} = -\sin\theta\cos\theta$, $\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}$, $\Gamma^3_{32} = \cot\theta$;
$R_{00} = -3\,\frac{\ddot{a}}{c^2a}$, $R_{11} = \frac{1}{c^2}\,\frac{a\ddot{a}+2\dot{a}^2+2c^2k}{1-kr^2}$, $R_{22} = \frac{r^2}{c^2}\left(a\ddot{a} + 2\dot{a}^2 + 2c^2k\right)$, $R_{33} = R_{22}\sin^2\theta$, $R = -\frac{6}{c^2a^2}\left(a\ddot{a} + \dot{a}^2 + c^2k\right)$

Friedmann eqs.: $H^2 = \frac{\dot{a}^2}{a^2} = \frac{8}{3}\,\pi G\rho + \frac{c^2}{3}\,\Lambda - \frac{c^2}{a^2}\,k$, $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3}{c^2}\,p\right) + \frac{c^2}{3}\,\Lambda$; eq. of state $w = p/\rho c^2$

Schwarzschild: $\mathrm{d}s^2 = -\big(1 - \frac{2GM}{r}\big)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-\frac{2GM}{r}} + r^2\,\mathrm{d}\Omega^2$, contains two symmetry assumptions: sphericity & static; asymptotically flat for $r \to \infty$ or $M \to 0$; g_{tt} and g_{rr} switch sign at $r = r_s = \frac{2GM}{c^2} \Rightarrow$ light cones flip and are undefined at r_s;
$\Gamma^1_{00} = \frac{GM}{r^3}\,(r - 2GM)$, $\Gamma^1_{11} = \frac{-GM}{r(r-2GM)} = -\Gamma^0_{01}$, $\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}$, $\Gamma^1_{22} = -r\big(1 - \frac{2GM}{r}\big)$, $\Gamma^1_{33} = \Gamma^1_{22}\sin^2\theta$, $\Gamma^2_{33} = -\sin\theta\cos\theta$, $\Gamma^3_{23} = \cot\theta$;
$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{12r_s^2}{r^6}$ (while $R = 0$) ⟶ curvature finite except at $r = 0$; in Kruskal coord. (u, v, θ, φ): $\mathrm{d}s^2 = \frac{4r_s^3}{r}\,e^{-\frac{r}{2GM}}\left(\mathrm{d}v^2 - \mathrm{d}u^2\right) + r^2\,\mathrm{d}\Omega^2$

Linearized gravity: $\Box\overline{h}^{\mu\nu} = -\frac{16\pi G}{c^4}T^{\mu\nu}$, gauge: $\partial_\nu\overline{h}^{\mu\nu} = 0$, $\overline{h}^{\mu\nu} = h^{\mu\nu} - \frac{h}{2}\eta^{\mu\nu}$ (harmonic gauge condition $\partial_\mu\overline{h}^\mu_\nu = 0$); in vacuum: $\Box\overline{h}^{\mu\nu} = 0$ solved by plane wave $\overline{h}^{\mu\nu} = A^{\mu\nu}e^{\pm ik_\sigma x^\sigma}$ with $k^2 = 0$; traceless transverse gauge fixes all gauge freedom, leaves 2 physical d.o.f., $\overline{h}^{\mathrm{TT}}_{\mu\nu} = h^{\mathrm{TT}}_{\mu\nu}$, $A^{\mu\nu}_{\mathrm{TT}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b & -a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Lie derivative of V w.r.t. vector field a: $\mathcal{L}_a V^\mu = a^\nu\partial_\nu V^\mu - V^\nu\partial_\nu a^\mu$, $\mathcal{L}_a V_\mu = a^\nu\partial_\nu V_\mu + V_\nu\partial_\mu a^\nu$; $\mathcal{L}_a g_{\mu\nu} = \nabla_\nu a_\mu + \nabla_\mu a_\nu$ for metric compatible connection; if $\mathcal{L}_\xi g_{\mu\nu} = 0$, then ξ is Killing vector field

Killing vector fields are vector fields on a Riemannian manifold that preserve the metric. They are the infinitesimal generators of isometries which in turn generate symmetries.

Useful formulae from exercises

- harmonic oscillator $m\ddot{x} = -kx$ with angular frequency $\omega = \sqrt{\frac{k}{m}} = \frac{2\pi}{T}$ and period T, $x(t) = A\cos(\omega t + \varphi)$, potential energy $U = \frac{1}{2}kx^2(t)$
- gravit. force $F_g = -\frac{Gm_1m_2}{r^2}$, centrifugal force $F_c = \frac{m_2v^2}{r}$; in stable orbit $-F_g = F_c \Longrightarrow$ orbital velocity $v = \sqrt{\frac{Gm_1}{r}}$
- e.o.m. invariant under $L(q,\dot{q},t) \longrightarrow \alpha L(q,\dot{q},t) + \frac{\mathrm{d}}{\mathrm{d}t}f(q,t)$, this yields symmetries and conservation laws, $\frac{\partial L}{\partial q_i} = 0 \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = \frac{\mathrm{d}}{\mathrm{d}t}p_i = 0$
- escape velocity follows from $T + V = \frac{m}{2}v_{\mathrm{esc}}^2 - \frac{GmM}{r} = 0 \Longrightarrow v_{\mathrm{esc}} = \sqrt{\frac{2GM}{r}}$, for $v_{\mathrm{esc}} = c$, we get $R_s = \frac{2GM}{c^2}$
- $\eta_{\mu\nu} = \Lambda^\rho_\mu\Lambda^\sigma_\nu\eta_{\rho\sigma}$ follows from requiring $|x'|^2 = |x|^2$
- relativistic velocity addition: $w = \frac{u+v}{1+\frac{uv}{c^2}}$; proper velocity $v = \frac{s}{\tau} = \frac{\frac{1}{\gamma}d}{\tau} = \sqrt{1-\frac{v^2}{c^2}}\,\frac{d}{\tau}$, where e.g. $\frac{d}{\tau} = c$ for $d = x$ ly and $\tau = x$ years; relativistic Doppler effect: observer moving away with v measures frequency $f' = \sqrt{\frac{1-\beta}{1+\beta}}f_0$; two events separated by timelike (spacelike) intervals, i.e. $\Delta s^2 > (<)0$, may have (may not have) cause-effect relationship, $\exists$ frame s.t. events at same location (time)
- weak field limit: holds for slowly moving particles ($v \ll c$) and weak gravitational fields $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$ (that are static, i.e. $\partial_0 g_{\mu\nu} = 0$)
- Christoffel symbols in spherical coord.: $\Gamma^1_{22} = -r$, $\Gamma^1_{33} = -r\sin^2\theta$, $\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}$, $\Gamma^2_{33} = -\sin\theta\cos\theta$, $\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}$, $\Gamma^3_{23} = \Gamma^3_{32} = \cot\theta$, all others zero
- effective potential for orbiting objects $V_{\mathrm{eff}}(r) = -\frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{c^2r^3}$, where last term is a pure GR contribution
- particles experiencing grav. waves don't change their position because of them
- Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- coordinate transformations:
  Cartesian $\leftrightarrow$ spherical $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos\varphi\sin\theta \\ r\sin\varphi\sin\theta \\ r\cos\theta \end{pmatrix}$, $\begin{pmatrix} r \\ \theta \\ \varphi \end{pmatrix} = \begin{pmatrix} \sqrt{x^2+y^2+z^2} \\ \arccos\left(z/\sqrt{x^2+y^2+z^2}\right) \\ \arctan(y/x) \end{pmatrix}$;
  Cartesian $\leftrightarrow$ cylindrical $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \rho\cos\varphi \\ \rho\sin\varphi \\ z \end{pmatrix}$, $\begin{pmatrix} \rho \\ \varphi \\ z \end{pmatrix} = \begin{pmatrix} \sqrt{x^2+y^2} \\ \arcsin(y/\rho) \\ z \end{pmatrix}$;
  Cartesian $\leftrightarrow$ ellipsoidal $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (r^2+a^2)^{1/2}\sin\theta\cos\varphi \\ (r^2+a^2)^{1/2}\sin\theta\sin\varphi \\ r\cos\theta \end{pmatrix}$;
  Cartesian $\leftrightarrow$ ellipsoid $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\cosh\mu\cos\nu \\ a\sinh\mu\sin\nu \end{pmatrix}$;
  Cartesian $\leftrightarrow$ hyperbolic $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ve^u \\ ve^{-u} \end{pmatrix}$, $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\ln(x/y) \\ \sqrt{xy} \end{pmatrix}$
- Tensor $\underline{T}$ of rank (k,l) is a multilinear map that takes k dual vectors and l vectors and projects them onto a number in $\mathbb{R}$.
  Expansion: $\underline{T} = T^{\mu_1\cdots\mu_k}_{\nu_1,\ldots\nu_l'}\partial_{\mu_1}\otimes\cdots\otimes\partial_{\mu_k'}\otimes\mathrm{d}x^{\nu_1}\otimes\cdots\otimes\mathrm{d}x^{\nu_l}$.
  Transformation: $T^{\mu_1\cdots\mu_k}_{\nu_1'\cdots\nu_l'} = \frac{\partial x^{\mu_1'}}{\partial x^{\mu_1}}\cdots\frac{\partial x^{\mu_k'}}{\partial x^{\mu_k}}\frac{\partial x^{\nu_1}}{\partial x^{\nu_1'}}\cdots\frac{\partial x^{\nu_l}}{\partial x^{\nu_l'}}T^{\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_l}$
  Christoffel symbol transformation $\Gamma^{\alpha'}_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}\frac{\partial x^{\alpha'}}{\partial x^\alpha}\Gamma^\alpha_{\mu\nu} - \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}\frac{\partial^2 x^{\alpha'}}{\partial x^\mu\partial x^\nu}$