Quantum Field Theory II – Exam Sheet

Advanced Quantum Field Theory

Exam cheat sheet by Janosh Riebesell

Path Integral

- QM transition amplitude: $\langle q_F, t_F \mid q_I, t_I\rangle = \langle q_F \mid e^{-i\hat{H}\overbrace{(t_F - t_I)}^{\Delta t}} \mid q_I\rangle$, $\delta t = \frac{\Delta t}{N+1}$, $N \to \infty$, $e^{-i\hat{H}\Delta t} = e^{-i\hat{H}\delta t}\cdots e^{-i\hat{H}\delta t}$. Insert $\mathbb{1} = \int \mathrm{d}q_k \, |q_k\rangle\langle q_k|$ to get

$$\langle q_F, t_F \mid q_I, t_I\rangle = \lim_{N\to\infty} \int \prod_{k=1}^{N} \mathrm{d}q_k \langle q_F \mid e^{-i\hat{H}\delta t} \mid q_N\rangle\langle q_N \mid e^{-i\hat{H}\delta t} \mid q_{N-1}\rangle \cdots \langle q_1 \mid e^{-i\hat{H}\delta t} \mid q_I\rangle,$$

  now assume $\hat{H}(\hat{p},\hat{q}) = f(\hat{p}) + V(\hat{q})$ and use BCH formula $e^{-i\hat{H}\delta t} = e^{-iV(\hat{q})\delta t} e^{-if(\hat{p})\delta t} + \mathcal{O}(\delta t^2)$, thus $\langle q_{k+1} \mid e^{-i\hat{H}\delta t} \mid q_k\rangle = \int \mathrm{d}p_k \langle q_{k+1} \mid e^{-if(\hat{p})\delta t} \mid p_k\rangle\langle p_k \mid e^{-iV(\hat{q})\delta t} \mid q_k\rangle$, where $\langle p_k \mid q_k\rangle = \frac{e^{-ip_k q_k}}{\sqrt{2\pi}}$ so that $\int \frac{\mathrm{d}p_k}{2\pi} e^{-i(f(p_k)+V(q_k))\delta t} e^{ip_k(q_{k+1}-q_k)}$. Reinsertion gives

$$\langle q_F, t_F \mid q_I, t_I\rangle = \lim_{N\to\infty} \int \frac{\mathrm{d}p_0}{2\pi} \prod_{k=1}^{N} \frac{\mathrm{d}p_k \, \mathrm{d}q_k}{2\pi} e^{i\sum_{k=0}^{N}\left[p_k \frac{q_{k+1}-q_k}{\delta t} - H(p_k,q_k)\right]\delta t}$$

$$\xrightarrow{\delta t\to 0} \int_{q_I}^{q_F} \mathcal{D}q(t)\mathcal{D}p(t) e^{i\int_{t_I}^{t_F}(p\dot{q} - H(\hat{p},\hat{q}))\,\mathrm{d}t} = \int_{q_I}^{q_F} \mathcal{D}q(t)\mathcal{D}p(t) \underbrace{e^{i\int_{t_I}^{t_F}\mathrm{d}t\, L(p,q)}}_{e^{iS[p,q]}}$$

- Scalar fields: path integral master formula for quantum correlation fcts.

$$G(x_1,...,x_n) = \langle\Omega \mid T\prod_{i=1}^{n}\hat{\varphi}_H(x_i) \mid \Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{\int \mathcal{D}\varphi(x)\prod_{i=1}^{n}\varphi(x_i) e^{i\int_{-T}^{T}\mathrm{d}^4x\,\mathcal{L}(\varphi(x))}}{\int \mathcal{D}\varphi(x) e^{i\int_{-T}^{T}\mathrm{d}^4x\,\mathcal{L}(\varphi(x))}}$$

- $Z[J]$ generating functional of Green's fcts.: $G(x_1,...,x_n) = \frac{1}{Z[0]}\prod_{i=1}^{n}\frac{\delta}{i\delta J(x_i)} Z[J]|_{J=0}$; $Z_0[J] = \int \mathcal{D}\varphi e^{-\frac{1}{2}\varphi\cdot D_F^{-1}\cdot\varphi + i\varphi\cdot J}$
- Schwinger–Dyson eq. $\int \mathcal{D}\varphi\Big(\frac{\delta S}{\delta\varphi(x)} + J(x)\Big)e^{i(S[\varphi]+J\cdot\varphi)} = 0$, states that cl. e.o.m. holds as operator eq. in quantum theory; $\left(\frac{\delta S}{\delta\varphi}\Big|_{\varphi=\frac{\delta}{i\delta J}} + J\right)Z[J] = 0$
- $iW[J] = \ln\frac{Z[J]}{Z[0]}$ generating fctnal of fully connected Green's fcts.
- 1PI effective action $\Gamma[\phi]$ Legendre transform of $W[J]$, i.e. $\Gamma[\phi] \coloneqq W\big[J_\phi\big] - \phi\cdot J_\phi$; $\frac{\delta\Gamma}{\delta\phi(x)} = -J_\phi(x)$, $G_2^{(c)}(x_1,x_2) = -\Gamma_2^{-1}(x_1,x_2)$
- fermionic path integral: $\langle\Psi_F(\vec{x}_F), t_F \mid \Psi_I(\vec{x}_I), t_I\rangle = \int \mathcal{D}\overline{\Psi}(x)\mathcal{D}\Psi(x) e^{i\int_{t_I}^{t_F}\mathrm{d}^4x\,\mathcal{L}(\Psi,\overline{\Psi})}$
- if $H(p,q) = \frac{p^2}{2m} + V(q)$, integrate $\int \mathcal{D}p$ explicitly by analytic continuation $\delta t \to \delta t(1-i\varepsilon)$ (so that $\mathrm{Re}(a) > 0$) to get

$$\int \frac{\mathrm{d}p_k}{2\pi} e^{i\left(p_k(q_{k+1}-q_k) - \delta t\frac{p_k^2}{2m}\right)} = e^{i\frac{m}{2\delta t}(q_{j+1}-q_j)^2}\sqrt{\frac{-im}{2\pi\delta t}}$$

- ex. of compl. the square in fermionic path integral:

$$S_0\big[\overline{\Psi},\Psi\big] + \overline{\eta}\cdot\Psi + \overline{\Psi}\cdot\eta = i\overline{\Psi}\cdot S_F^{-1}\cdot\Psi + \overline{\eta}\cdot\Psi + \overline{\Psi}\cdot\eta = \Big(\overline{\Psi} - iS_F\overline{\eta}\Big)\cdot iS_F^{-1}\cdot(\Psi - iS_F\eta) + i\overline{\eta}\cdot S_F\cdot\eta$$

Renormalization

- Superficial degree of divergence D: φ^n in d dim $D_{\varphi^n} = d - \overbrace{\big(d - n\frac{d-2}{2}\big)}^{[\lambda^{(n)}]}V - \frac{d-2}{2}E$ with $nV = 2I + E$ and Euler's formula $L = I - V + 1$ or $L = \sum_i I_i - \sum_j V_j + 1$
- $D \geq 0$ may still be finite due to symmetries, $D < 0$ may still be divergent due to divergent subdiagram; tree-level diagrams have $D = 0$ but are finite
- a QFT is 1. renormalizable if number of superf. diverg. ampl. is finite, but superf. divergs. appear at every order; 2. super-renormalizable if only finitely many diagrams are superf. divergent; 3. non-renormalizable if infinite; 1. $\hat{=}$ $[\lambda] = 0$, 2. $\hat{=}$ $[\lambda] > 0$, 3. $\hat{=}$ $[\lambda] < 0$
- BPHZ theorem: if a QFT is renormalizable, i.e. has finite number of divergent diagr., one can absorb divergencies order by order in counterterms
- in a QFT with dimensionless coupling, $[\lambda] = 0$, the dimension of an amplitude is equal to D
- QED: $D_{\mathrm{QED}} = 4 - E_\gamma - \frac{3}{2}E_f$, QED is renormalizable since $[e] = 0$, symmetric under charge conj.!, respects Ward identity!
- Callan–Symanzik eq.: $\mu\frac{\mathrm{d}}{\mathrm{d}\mu}G_n^{(0)}(x;\lambda_0,m_0)|_{\lambda_0,m_0\text{ fixed}} = 0 = \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\big(Z^{n/2}G_n(x;\lambda,m;\mu)\big)|_{\lambda_0,m_0\text{ fixed}}$, where $G_n(x_1,...,x_n) = \langle\Omega \mid T\prod_{i=1}^{n}\varphi(x_i) \mid \Omega\rangle|_{\text{connected}}$ and $\varphi(x_i) = Z^{-1/2}\varphi_0(x_i)$; chain rule yields

$$\left(\mu\frac{\partial}{\partial\mu} + \beta_\lambda\frac{\partial}{\partial\lambda} + \beta_{m^2}\frac{\partial}{\partial m^2} + n\cdot\gamma_\varphi\right)G_n(x;\lambda,m;\mu) = 0$$

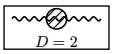
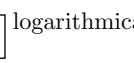
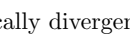
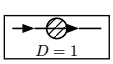
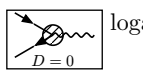
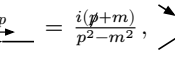
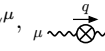
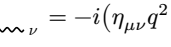
  with $\beta_\lambda \coloneqq \mu\frac{\mathrm{d}\lambda}{\mathrm{d}\mu}\Big|_{\lambda_0,m_0}$, $\beta_{m^2} = \mu\frac{\mathrm{d}m^2}{\mathrm{d}\mu}\Big|_{\lambda_0,m_0}$, $\gamma_\varphi = \frac{1}{2}\frac{\mu}{Z}\frac{\mathrm{d}Z}{\mathrm{d}\mu}\Big|_{\lambda_0,m_0}$

- Renormalization group (RG) flow (eq.) given by the beta function $\beta(\lambda) = \frac{\mathrm{d}\lambda(\mu)}{\mathrm{d}\ln(\mu)}$ describes change of $\lambda(\mu)$ with μ: $\beta(\lambda) > 0\,(<0) \iff \lambda(\mu)$ increases (decreases) as μ increases. Landau pole: $\lambda \to \infty$ as $\mu \to \mu^*$ with μ^* finite; Gaussian IR (UV) fixed point: $\lambda \to 0$ as $\mu \to 0\,(\infty)$; $\beta = 0\,\forall\mu \Rightarrow \lambda(\mu) \longleftrightarrow$ theory is scale-ind. or conformal
- RG flow of dimensionful operators: $\int \mathrm{d}^d x\, C_i \mathcal{O}_{d_i}$ with $\big[\mathcal{O}_{d_i}\big] = d_i$, $[C_i] = d - d_i$, $C_i = g_i \mu^{d-d_i}$, then CS eq. reads
$$0 = \bigg[\mu\frac{\partial}{\partial\mu} + \beta_\lambda\frac{\partial}{\partial\lambda} + n\gamma_\varphi + \underbrace{(\gamma_i + d_i - d)g_i}_{\beta_i := \frac{\mathrm{d}g_i}{\mathrm{d}\ln\mu}}\frac{\partial}{\partial g_i}\bigg] G_n(x;\lambda,g_i,\mu)$$
if $[C_i] = d - d_i < 0\,(>0) \longleftrightarrow C_i$ is non-renormalizable (super-renormalizable) coupling which becomes irrelevant (relevant) in the IR
- Wilsonian approach: QFT just an effective description of physics accurate only at energies below an intrinsic cutoff Λ_0 (or at large distances)
- Dim. reg. integrals:
$$\int\frac{\mathrm{d}^d l}{(2\pi)^d}\,\frac{1}{(l^2-\Delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}}\,\frac{\Gamma\big(n-\frac{d}{2}\big)}{\Gamma(n)}\,\Delta^{\frac{d}{2}-n},\quad \int\frac{\mathrm{d}^d l}{(2\pi)^d}\,\frac{l^2}{(l^2-\Delta)^n} = \frac{(-1)^{n-1} i}{(4\pi)^{d/2}}\,\frac{d}{2}\,\frac{\Gamma\big(n-\frac{d}{2}-1\big)}{\Gamma(n)}\,\Delta^{1+\frac{d}{2}-n},$$
$$\int\frac{\mathrm{d}^d l}{(2\pi)^d}\,\frac{l^\mu l^\nu}{(l^2-\Delta)^n} = \frac{(-1)^{n-1} i}{(4\pi)^{d/2}}\,\frac{\eta^{\mu\nu}}{2}\,\frac{\Gamma\big(n-\frac{d}{2}-1\big)}{\Gamma(n)}\,\Delta^{1+\frac{d}{2}-n}$$
frequent identity: $\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}}\,\Delta^{\frac{d}{2}-2} = \frac{1}{16\pi^2}\big(\frac{2}{\varepsilon} - \ln\Delta - \gamma + \ln 4\pi + \mathcal{O}(\varepsilon)\big)$,
$$\frac{1}{A_1^{m_1}A_2^{m_2}\cdots A_n^{m_n}} = \int_0^1 \mathrm{d}x_1\cdots\mathrm{d}x_n\,\delta\bigg(\sum_i x_i - 1\bigg)\,\frac{\prod x_i^{m_i-1}}{\big[\sum_i x_i A_i\big]^{\sum m_i}}\,\frac{\Gamma(m_1+\cdots+m_n)}{\Gamma(m_1)\cdots\Gamma(m_n)},\quad \frac{1}{AB^n} = \int_0^1 \mathrm{d}x\,\frac{n(1-x)^{n-1}}{[xA+(1-x)B]^{n+1}}$$
- Trace identities: $\mathrm{tr}(\gamma^\mu) = 0$, $\mathrm{tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_n}) = 0$ for n odd, $\mathrm{tr}(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu}$, $\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho})$, $\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu$, $\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4\eta^{\nu\rho}$

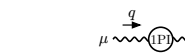
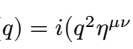
Yang–Mills theory

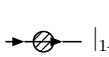
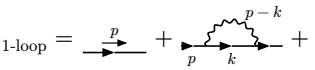
- starting point: Lie group H with Lie algebra $\mathrm{Lie}(H)$ s.t. every $h \in H$ can be expressed as $h = e^{-ig\alpha}$ with $g \in \mathbb{R}$, $\alpha = \alpha_a T^a \in \mathrm{Lie}(H)$, where T^a are a basis of $\mathrm{Lie}(H)$, i.e. the generators of H, satisfying defining relation $[T^a, T^b] = if^{abc}T^c$ and Jacobi identity $[[T^a,T^b],T^c] + \text{c.p.} = 0$
- (adjoint) covariant derivative: $D_\mu\alpha(x) := \partial_\mu\alpha(x) + ig\big[A_\mu(x),\alpha(x)\big]$ used to define $F_{\mu\nu}(x) = \frac{1}{ig}\big[D_\mu, D_\nu\big] = \partial_\mu A_\nu - \partial_\nu A_\mu + ig\big[A_\mu, A_\nu\big]$, field strength satisfies Bianchi identity $D_\mu F_{\alpha\beta} + D_\alpha F_{\beta\mu} + D_\beta F_{\mu\alpha} = 0$
- pure Yang–Mills Lagrangian: $\mathcal{L}_{\mathrm{YM}} = -\frac{1}{2}\,\mathrm{tr}\big(F_{\mu\nu}F^{\mu\nu}\big) = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a}$ is gauge invariant; e.o.m. for A_μ: $D_\mu F^{\mu\nu} = 0 = \partial_\mu F^{\mu\nu} + ig\big[A_\mu, F^{\mu\nu}\big]$
- generators T^a normalized to $\mathrm{tr}\big(T^aT^b\big) = \frac{1}{2}\,\delta^{ab}$ (used in the second step of $\mathcal{L}_{\mathrm{YM}}$)
- for $\mathcal{L}_{\mathrm{YM}} + \mathcal{L}_{\mathrm{matter}}$, the e.o.m. become $D_\mu F^{\mu\nu a} = j^{\nu a}$ for some $j^{\nu a}(\varphi,\Psi,\ldots)$

Possible exam questions

1. Derivation of the fermionic PI: trans. ampl. $\langle\Psi_F,t_F \mid \Psi_I,t_I\rangle = \langle\Psi_F \mid e^{-i\hat H(t_F-t_I)} \mid \Psi_I\rangle$ with $e^{-i\hat H(t_F-t_I)} = \lim_{N\to\infty}\big(e^{-i\hat H\delta t}\big)^N$, $\delta t = \frac{t_F-t_I}{N+1}$. Now insert identities
$$\langle\Psi_F,t_F \mid \Psi_I,t_I\rangle = \lim_{N\to\infty}\int\prod_{j=1}^N \mathrm{d}\Psi_j^*\,\mathrm{d}\Psi_j \langle\Psi_F \mid e^{-i\hat H\delta t} \mid \Psi_N\rangle\cdots\langle\Psi_1 \mid e^{-i\hat H\delta t} \mid \Psi_I\rangle$$
since $\mathbb{1} = \int \mathrm{d}\Psi_j^*\,\mathrm{d}\Psi_j\,|\Psi_j\rangle e^{-\Psi_j^*\Psi_j}\langle\Psi_j|$. We assume $\hat H = \hat\Psi^\dagger M\hat\Psi$ with Grassmann-even M. Then
$$\langle\Psi_{j+1} \mid e^{-i\hat H\delta t} \mid \Psi_j\rangle = \langle\Psi_{j+1} \mid \big(1 - i\hat\Psi^\dagger M\hat\Psi\delta t\big) \mid \Psi_j\rangle = e^{\Psi_{j+1}^*\Psi_j}e^{-i\Psi_{j+1}^*M\Psi_j\delta t} = e^{\Psi_{j+1}^*\Psi_{j+1}}e^{\Psi_{j+1}^*(\Psi_j-\Psi_{j+1})}e^{-i\Psi_{j+1}^*M\Psi_j\delta t},$$
where the factors $e^{\Psi_{j+1}^*\Psi_{j+1}}$ cancel with those from identities, except for $e^{\Psi_F^*\Psi_F}$. Thus
$$\langle\Psi_F,t_F \mid \Psi_I,t_I\rangle = \lim_{N\to\infty}\int\prod_{j=1}^N \mathrm{d}\Psi_j^*\,\mathrm{d}\Psi_j e^{\Psi_F^*\Psi_F}e^{\sum_{j=0}^N(-\Psi_{j+1}^*(\Psi_{j+1}-\Psi_j)-i\delta t H(\Psi_{j+1}^*,\Psi_j))}$$
$$\xrightarrow{N\to\infty}\int\mathcal{D}\Psi^*\mathcal{D}\Psi e^{i\int_{t_I}^{t_F}\mathrm{d}t\,\left(\Psi^* i\frac{\partial\Psi}{\partial t} - H\right)}.$$
2. Renormalization of QED: $D = 4L - P_e - 2P_\gamma$, $L = P_e + P_\gamma - V + 1$, $V = 2P_\gamma + I_\gamma = \frac{1}{2}(2P_e + I_e) \Longrightarrow D = 4 - \frac{3}{2}I_e - I_\gamma$.
Divergent amplitudes (superficially):  $D=4$ can be absorbed into V_0,  $D=3$ zero by $A_\mu \to -A_\mu$ symmetry,
 $D=2$ logarithmically divergent,  $D=1$ 0 by $A_\mu \to -A_\mu$,  $D=0$ divergent parts cancel due to Ward identity,
 0 by charge conserv.,  $D=1$ logarithm. divergent,  $D=0$ logarithm. divergent,
$$\mathcal{L}_{\mathrm{QED}} = -\frac{1}{4}\big(F_{\mu\nu}\big)^2 + \overline\Psi(i\partial\!\!\!/ - m)\Psi - eA_\mu\overline\Psi\gamma^\mu\Psi - \frac{1}{4}\delta_A\big(F_{\mu\nu}\big)^2 + \overline\Psi(i\delta_\Psi\partial\!\!\!/ - \delta_m)\Psi - \delta_e A_\mu\overline\Psi\gamma^\mu\Psi,$$
where $\Psi = Z_\Psi^{-1/2}\Psi_0$, etc.
Feynman rules:  $= \frac{-i\eta_{\mu\nu}}{q^2}$, $= \frac{i(\not p + m)}{p^2-m^2}$,  $= -ie\gamma^\mu$,  $= -i\big(\eta_{\mu\nu}q^2 - q_\mu q_\nu\big)\delta_A$,

 $= i(\not{p}\delta_\Psi - \delta_m)$,  $= -ie\gamma^\mu\delta_e$

Renormalization conditions: $\Sigma(\not{p}=m)=0$, $\frac{\mathrm{d}}{\mathrm{d}\not{p}}\Sigma(\not{p}=m)=0$, $\Pi(q^2=0)=0$, $-ie\Gamma^\mu(p'-p=0)=-ie\gamma^\mu$, where  $= i\Pi^{\mu\nu}(q) = i(q^2\eta^{\mu\nu}-q^\mu q^\nu)\Pi(q^2)$ and $\left($  $\right)_{\text{amp}} = -ie\Gamma^\mu(p',p)$. We use dim. reg. to control UV and photon mass μ to control IR divergencies.

 $|_{\text{1-loop}} =$  $= \dfrac{i(\not{p}+m)}{p^2-m^2} - i\Sigma_2(p) + i(\not{p}\delta_\Psi - \delta_m)$

$$-i\Sigma_2(p) = (-ie)^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\gamma^\mu\frac{i(\not{k}+m)}{k^2-m^2}\gamma_\mu\frac{-i}{(p-k)^2-\mu^2} = -e^2\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^dl}{(2\pi)^d}\frac{-2x\not{p}+4m}{\left[l^2-\Delta\right]^2}$$

$$= \frac{-ie^2}{(4\pi)^{d/2}}\int_0^1\mathrm{d}x\,\frac{\Gamma(\frac{\varepsilon}{2})[(4-\varepsilon)m-2(1-\frac{\varepsilon}{2})x\not{p}]}{[(1-x)m^2+x\mu^2-x(1-x)p^2]^{\varepsilon/2}}$$

$$\Sigma(\not{p}=m)|_{\text{1-loop}} = -i\Sigma_2(m) + i(m\delta_\Psi-\delta_m) \overset{!}{=} 0 \Longrightarrow m\delta_\Psi-\delta_m = \Sigma_2(m) = \frac{e^2m}{(4\pi)^{d/2}}\int_0^1\mathrm{d}x\,\frac{\Gamma(\frac{\varepsilon}{2})(4-2x-\varepsilon(1-x))}{[(1-x)^2m^2+x\mu^2]^{\varepsilon/2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}\not{p}}\Sigma(\not{p}=m)|_{\text{1-loop}} = -i\frac{\mathrm{d}}{\mathrm{d}\not{p}}\Sigma_2(m) + i\delta_\Psi \overset{!}{=} 0$$

$$\Longrightarrow \delta_\Psi = -\frac{e^2}{(4\pi)^{d/2}}\int_0^1\mathrm{d}x\,\frac{\Gamma(\varepsilon/2)}{[(1-x)^2m^2+x\mu^2]^{\varepsilon/2}}\left[(2-\varepsilon)x-\frac{\varepsilon}{2}\frac{2x(1-x)m^2}{(1-x)^2m^2+x\mu^2}(4-2x-\varepsilon(1-x))\right]$$