

General Relativity – Exam Sheet

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Newton's axioms: 1. free particles move along straight lines at constant velocity, 2. $\vec{F} = m\ddot{\vec{x}}$, 3. actio = –reactio

Galilei transformations: universal time $t = t' \forall$ observers; 2 frames S, S' in relative motion $\parallel$ x -axis: $x'_1 = x_1 - vt$, $x'_2 = x_2$, $x'_3 = x_3$,
 $u'_1 = \frac{dx'_1}{dt} = u_1 - v$; $a'_1 = \frac{du'_1}{dt} = a_1 \implies$ accelerations identical in all frames, particles free in S are free in S'

Lorentz transformations: $t \neq t'$, based on two assumptions: 1. spacetime homogeneous, 2. $c = c' \forall$ observers. Classification of events valid in all frames: $ds^2 > 0$ timelike, $ds^2 = 0$ lightlike, $ds^2 < 0$ spacelike

Proper time τ displayed by comoving clock; $ds^2 = c^2 d\tau^2$, $d\tau = \frac{1}{\gamma} dt$

(Weak) equivalence principle: $m_i = m_g$, i.e. inertial and gravitational mass of any object are equal

Length of a curve C parametrized by λ : $I = \int_C ds = \int_C \sqrt{|g_{\mu\nu} dx^\mu dx^\nu|} = \int_a^b d\lambda \sqrt{|g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}|}$, area: $dA = \sqrt{|g_{\mu\nu}|} dx^\mu dx^\nu \stackrel{2d}{=} \sqrt{|g_{11}g_{22}|} dx^1 dx^2$

Riemannian manifolds are locally Cartesian, i.e. can locally be matched to Minkowski space

Christoffel symbols (affine connection): $\Gamma_{\mu\nu}^\rho = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$, can be obtained by diff. $g_{\mu\nu} = \vec{e}_\mu \cdot \vec{e}_\nu$ w.r.t. x^ρ and cyclically perm. indices; $\Gamma_{\mu\nu}^\rho = \vec{e}^\rho \cdot \frac{\partial}{\partial x^\nu} \vec{e}_\mu$; equivalence: locally Cartesian $\leftrightarrow$ vanishing connection; torsion $T_{\mu\nu}^\alpha := \Gamma_{\mu\nu}^\alpha - \Gamma_{\nu\mu}^\alpha$

Cov. derivative of contravector $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma_{\mu\rho}^\nu V^\rho$, of covector $\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma_{\mu\nu}^\rho V_\rho$; $\nabla_\mu V^\nu = \partial_\mu V^\nu$ if locally Cartesian

Geodesic (on torsion-free manifold): shortest connection of two points, more generally curve with constant tangent. Geodesic equation (eq. of parallel transport): $\frac{d^2 x^\mu}{d\lambda^2} + \Gamma_{\rho\sigma}^\mu \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0 = \frac{du^\mu}{d\lambda} + \Gamma_{\rho\sigma}^\mu u^\rho u^\sigma$, path-dependent due to varying $\Gamma_{\rho\sigma}^\mu$

Riemann tensor $R_{\sigma\mu\nu}^\rho = \partial_\mu \Gamma_{\nu\sigma}^\rho - \partial_\nu \Gamma_{\mu\sigma}^\rho + \Gamma_{\mu\lambda}^\rho \Gamma_{\nu\sigma}^\lambda - \Gamma_{\nu\lambda}^\rho \Gamma_{\mu\sigma}^\lambda$, $R_{\rho\sigma\mu\nu} = g_{\rho\lambda} R_{\sigma\mu\nu}^\lambda$; symmetries: $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu} = R_{\sigma\rho\nu\mu}$, $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$, $R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0$, $d = \frac{n^2}{12}(n^2 - 1)$ indep. comp.; metric's comp. constant $\iff R_{\sigma\mu\nu}^\rho = 0 \iff$ space flat

Ricci tensor $R_{\mu\nu} = R_{\mu\lambda\nu}^\lambda = R_{\nu\mu}$, Ricci scalar $R = R_\mu^\mu = g^{\mu\nu} R_{\mu\nu}$

Bianchi identity $\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} = 0$; Riemann tensor on 2-sphere $R_{jkm}^i = \frac{1}{r^2} (\delta_k^i g_{jm} - \delta_m^i g_{jk})$

Energy-mom. tensor of perfect fluid ($\equiv$ isotropic in its rest frame) $T^{\mu\nu} = (\rho + \frac{p}{c^2}) u^\mu u^\nu - p \eta^{\mu\nu} = \text{diag}(\rho c^2, p, p, p)$

Einstein field eqs. $G_{\mu\nu} = R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$

FLRW: $ds^2 = c^2 dt^2 - a^2(t) \left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right]$, inv. under $k \rightarrow k/|k|$, $r \rightarrow \sqrt{|k|}r$, $a \rightarrow a/\sqrt{|k|}$; metric singular at $r = \pm 1$

- $k = -1 \leftrightarrow$ negative curvature $\leftrightarrow$ open universe $\leftrightarrow \rho < \rho_{\text{crit}} \leftrightarrow \Omega < 1$
- $k = 0 \leftrightarrow$ no curvature $\leftrightarrow$ flat universe $\leftrightarrow \rho = \rho_{\text{crit}} \leftrightarrow \Omega = 1$
- $k = 1 \leftrightarrow$ positive curvature $\leftrightarrow$ closed universe $\leftrightarrow \rho > \rho_{\text{crit}} \leftrightarrow \Omega > 1$

For $k = \pm 1 \rightarrow$ new radial coordinate $r = \sin(h)\chi$, $\frac{dr}{d\chi} = \sqrt{1 \mp r^2}$; $\rho_{\text{crit}} = \frac{3H^2}{8\pi G}$;

$\Gamma_{11}^1 = \frac{kr}{1-kr^2}$, $\Gamma_{11}^0 = \frac{a\dot{a}}{c(1-kr^2)}$, $\Gamma_{22}^0 = \frac{a\dot{a}r^2}{c}$, $\Gamma_{33}^0 = \frac{a\dot{a}r^2}{c} \sin^2 \theta$, $\Gamma_{01}^1 = \Gamma_{10}^1 = \Gamma_{02}^2 = \Gamma_{20}^2 = \Gamma_{03}^3 = \Gamma_{30}^3 = \frac{\dot{a}}{ca}$, $\Gamma_{22}^1 = -r(1 - kr^2)$, $\Gamma_{33}^1 = \Gamma_{22}^1 \sin^2 \theta$, $\Gamma_{33}^2 = -\sin \theta \cos \theta$, $\Gamma_{12}^2 = \Gamma_{21}^2 = \Gamma_{13}^3 = \Gamma_{31}^3 = \frac{1}{r}$, $\Gamma_{32}^3 = \cot \theta$;
 $R_{00} = 3 \frac{\ddot{a}}{c^2 a}$, $R_{11} = -\frac{1}{c^2} \frac{a\ddot{a} + 2\dot{a}^2 + 2c^2 k}{1-kr^2}$, $R_{22} = -\frac{r^2}{c^2} (a\ddot{a} + 2\dot{a}^2 + 2c^2 k)$, $R_{33} = R_{22} \sin^2 \theta$, $R = \frac{6}{c^2 a^2} (a\ddot{a} + \dot{a}^2 + c^2 k)$

Friedmann eqs.: $H^2 = \frac{\dot{a}^2}{a^2} = \frac{8}{3} \pi G \rho + \frac{c^2}{3} \Lambda - \frac{c^2}{a^2} k$, $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + \frac{3}{c^2} p) + \frac{c^2}{3} \Lambda$; eq. of state $w = p/\rho c^2$

Schwarzschild: $ds^2 = -(1 - \frac{2GM}{r}) dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2 d\Omega^2$, contains two symmetry assumptions: sphericity & static; asymptotically flat for $r \rightarrow \infty$ or $M \rightarrow 0$; g_{tt} and g_{rr} switch sign at $r = r_s = \frac{2GM}{c^2} \Rightarrow$ light cones flip and are undefined at r_s ;

$\Gamma_{00}^1 = \frac{GM}{r^3} (r - 2GM)$, $\Gamma_{11}^1 = \frac{-GM}{r(r-2GM)} = -\Gamma_{01}^0$, $\Gamma_{12}^2 = \Gamma_{21}^2 = \Gamma_{13}^3 = \Gamma_{31}^3 = \frac{1}{r}$, $\Gamma_{22}^1 = -r(1 - \frac{2GM}{r})$, $\Gamma_{33}^1 = \Gamma_{22}^1 \sin^2 \theta$,
 $\Gamma_{33}^2 = -\sin \theta \cos \theta$, $\Gamma_{23}^3 = \cot \theta$;

$R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \frac{3r_s^2}{r^6} R \rightarrow$ curvature finite except at $r = 0$; in Kruskal coord. (u, v, θ, φ) : $ds^2 = \frac{4r_s^3}{r} e^{-\frac{r}{2GM}} (dv^2 - du^2) + r^2 d\Omega^2$

Linearized gravity: $\square \bar{h}^{\mu\nu} = -\frac{16\pi G}{c^4} T^{\mu\nu}$, gauge: $\partial_\nu \bar{h}^{\mu\nu} = 0$, $\bar{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu}$ (harmonic gauge condition $\partial_\mu \bar{h}^\mu_\nu = 0$); in vacuum: $\square \bar{h}^{\mu\nu} = 0$ solved by plane wave $\bar{h}^{\mu\nu} = A^{\mu\nu} e^{\pm i k_\sigma x^\sigma}$ with $k^2 = 0$; traceless transverse gauge fixes all gauge freedom, leaves 2 physical d.o.f., $\bar{h}^{\text{TT}}_{\mu\nu} = h^{\text{TT}}_{\mu\nu}$, $A^{\mu\nu}_{\text{TT}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b & -a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Lie derivative of V w.r.t. vector field a : $\mathcal{L}_a V^\mu = -a^\nu \partial_\nu V^\mu + V^\nu \partial_\nu a^\mu$, $\mathcal{L}_a V_\mu = a^\nu \partial_\nu V_\mu + V_\nu \partial_\mu a^\nu$; $\mathcal{L}_a g_{\mu\nu} = \nabla_\nu a_\mu + \nabla_\mu a_\nu$ for metric compatible connection; if $\mathcal{L}_\xi g_{\mu\nu} = 0$, then ξ is Killing vector field

Killing vector fields are vector fields on a Riemannian manifold that preserve the metric. They are the infinitesimal generators of isometries which in turn generate symmetries.

Useful formulae from exercises

- harmonic oscillator $m\ddot{x} = -kx$ with angular frequency $\omega = \sqrt{\frac{k}{m}} = \frac{2\pi}{T}$ and period T , $x(t) = A \cos(\omega t + \varphi)$, potential energy $U = \frac{1}{2} kx^2(t)$
- gravit. force $F_g = -\frac{Gm_1 m_2}{r^2}$, centrifugal force $F_c = \frac{m_2 v^2}{r}$; in stable orbit $-F_g = F_c \Rightarrow$ orbital velocity $v = \sqrt{\frac{Gm_1}{r}}$
- e.o.m. invariant under $L(q, \dot{q}, t) \rightarrow \alpha L(q, \dot{q}, t) + \frac{d}{dt} f(q, t)$, this yields symmetries and conservation laws, $\frac{\partial L}{\partial q_i} = 0 \Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{d}{dt} p_i = 0$
- escape velocity follows from $T + V = \frac{m}{2} v_{\text{esc}}^2 - \frac{GmM}{r} = 0 \Rightarrow v_{\text{esc}} = \sqrt{\frac{2GM}{r}}$, for $v_{\text{esc}} = c$, we get $R_s = \frac{2GM}{c^2}$
- $\eta_{\mu\nu} = \Lambda^\rho_\mu \Lambda^\sigma_\nu \eta_{\rho\sigma}$ follows from requiring $|x'|^2 = |x|^2$
- relativistic velocity addition: $w = \frac{u+v}{1+\frac{uv}{c^2}}$; proper velocity $v = \frac{\delta}{\tau} = \frac{1}{\tau} \frac{d}{d\tau} = \sqrt{1 - \frac{v^2}{c^2}} \frac{d}{d\tau}$, where e.g. $\frac{d}{d\tau} = c$ for $d = x$ ly and $\tau = x$ years; relativistic Doppler effect: observer moving away with v measures frequency $f' = \sqrt{\frac{1-\beta}{1+\beta}} f_0$; two events separated by timelike (spacelike) intervals, i.e. $\Delta s^2 > (<) 0$, may have (may not have) cause-effect relationship, $\exists$ frame s.t. events at same location (time)
- weak field limit: holds for slowly moving particles ($v \ll c$) and weak gravitational fields $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$ (that are static, i.e. $\partial_0 g_{\mu\nu} = 0$)
- Christoffel symbols in spherical coord.: $\Gamma_{22}^1 = -r$, $\Gamma_{33}^1 = -r \sin^2 \theta$, $\Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{r}$, $\Gamma_{33}^2 = -\sin \theta \cos \theta$, $\Gamma_{13}^3 = \Gamma_{31}^3 = \frac{1}{r}$, $\Gamma_{23}^3 = \Gamma_{32}^3 = \cot \theta$, all others zero
- effective potential for orbiting objects $V_{\text{eff}}(r) = -\frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{c^2 r^3}$, where last term is a pure GR contribution
- particles experiencing grav. waves don't change their position because of them
- Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- coordinate transformations:
 - Cartesian $\leftrightarrow$ spherical $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \cos \varphi \sin \theta \\ r \sin \varphi \sin \theta \\ r \cos \theta \end{pmatrix}$, $\begin{pmatrix} r \\ \theta \\ \varphi \end{pmatrix} = \begin{pmatrix} \sqrt{x^2 + y^2 + z^2} \\ \arccos(z/\sqrt{x^2 + y^2 + z^2}) \\ \arctan(y/x) \end{pmatrix}$;
 - Cartesian $\leftrightarrow$ cylindrical $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \rho \cos \varphi \\ \rho \sin \varphi \\ z \end{pmatrix}$, $\begin{pmatrix} \rho \\ \varphi \\ z \end{pmatrix} = \begin{pmatrix} \sqrt{x^2 + y^2} \\ \arcsin(y/\rho) \\ z \end{pmatrix}$;
 - Cartesian $\leftrightarrow$ ellipsoidal $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (r^2 + a^2)^{1/2} \sin \theta \cos \varphi \\ (r^2 + a^2)^{1/2} \sin \theta \sin \varphi \\ r \cos \theta \end{pmatrix}$;
 - Cartesian $\leftrightarrow$ ellipsoid $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \cosh \mu \cos \nu \\ a \sinh \mu \sin \nu \end{pmatrix}$;
 - Cartesian $\leftrightarrow$ hyperbolic $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} v e^u \\ v e^{-u} \end{pmatrix}$, $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \ln(x/y) \\ \sqrt{xy} \end{pmatrix}$
- Tensor $\underline{T}$ of rank (k, l) is a multilinear map that takes k dual vectors and l vectors and projects them onto a number in $\mathbb{R}$.
 - Expansion: $\underline{T} = T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l} \partial_{\mu_1} \otimes \dots \otimes \partial_{\mu_k} \otimes dx^{\nu_1} \otimes \dots \otimes dx^{\nu_l}$.
 - Transformation: $T^{\mu'_1 \dots \mu'_k}_{\nu'_1 \dots \nu'_l} = \frac{\partial x^{\mu'_1}}{\partial x^{\mu_1}} \dots \frac{\partial x^{\mu'_k}}{\partial x^{\mu_k}} \frac{\partial x^{\nu_1}}{\partial x^{\nu'_1}} \dots \frac{\partial x^{\nu_l}}{\partial x^{\nu'_l}} T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l}$
 - Christoffel symbol transformation $\Gamma^{\alpha'}_{\mu' \nu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \Gamma^{\alpha}_{\mu \nu} - \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} \frac{\partial^2 x^{\alpha'}}{\partial x^{\mu} \partial x^{\nu}}$