31.01.2015

Exercises rel. for the exam

Sheet	1	2	3	4
1		✓		
2	✓	✓	✓	
3		✓	✓	
4	✓	✓		
5			✓	
6		✓		
7		✓	✓	
8				
9	✓	✓		✓
10			✓	
11	✓			

Revision of Exercises relevant for the Exam

Problem 1.2 (Classical Electrodynamics as a field theory)

From Classical Electrodynamics we recall the following form of the Lagrangian for the electromagnetic field in vacuum

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{3}$$

Here the field strength tensor $F^{\mu\nu}$ is defined as $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ in terms of the gauge potential A^μ.

a) Derive the Lorentz transformation property of $F^{\mu\nu}$ from the fact that A^μ is a 4-vector.

A^μ being a 4-vector implies that under a Lorentz transformation Λ it behaves like

$$A'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu A^\nu = \Lambda^\mu{}_\nu A^\nu.$$

The same holds for ∂^μ as can be seen from the following

$$\partial'^\mu := \frac{\partial}{\partial x'_\mu} = \frac{\partial x_\nu}{\partial x'_\mu}\frac{\partial}{\partial x_\nu} = \Lambda^\mu{}_\nu \frac{\partial}{\partial x_\nu} = \Lambda^\mu{}_\nu \partial^\nu,$$

where the Lorentz transformation comes from the fact that

$$(x')^2 = x'_\mu x'^\mu = x'_\mu \Lambda^\mu{}_\nu x^\nu \overset{!}{=} x_\mu x^\mu = x^2 \implies x_\mu = x'_\mu \Lambda^\mu{}_\nu.$$

Therefore, $F^{\mu\nu}$ transform like a 4-matrix, i.e.

$$F'^{\mu\nu} = \partial'^\mu A'^\nu - \partial'^\nu A'^\mu = \Lambda^\mu{}_\alpha \partial^\alpha \Lambda^\nu{}_\beta A^\beta - \Lambda^\nu{}_\beta \partial^\beta \Lambda^\mu{}_\alpha A^\alpha$$

$$= \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right) = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta F^{\alpha\beta}.$$

b) Derive the equations of motion from the general expression for the Euler-Lagrange equations.

$$\partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} = \frac{\partial \mathcal{L}}{\partial \phi}, \quad (4)$$

with ϕ replaced by A^μ.

We need to calculate two terms for which an expression of $\mathcal{L}$ in terms of the gauge potential A^μ will come in handy. We have

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) = -\frac{1}{4}(\partial^2 A^2 - \partial_\mu A^\mu \partial^\nu A_\nu - \partial^\mu A_\mu \partial_\nu A^\nu + \partial^2 A^2)$$

$$= \frac{1}{2}(\partial_\mu A^\mu \partial_\nu A^\nu - \partial^2 A^2)$$

$$\left.\begin{aligned} &\partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu A^\mu)} = \partial_\nu \frac{1}{2}(\partial_\mu A^\nu - \partial^\nu A_\mu) = \frac{1}{2}\partial_\nu F_\mu{}^\nu \\ &\frac{\partial \mathcal{L}}{\partial A^\mu} = 0 \end{aligned}\right\} \partial_\nu F_\mu{}^\nu = \partial^\nu F_{\mu\nu} = 0$$

Problem 2.1 (Canonical commutation relations)

Starting from the mode expansion of the Schrödinger fields,

$$\varphi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a(\vec{p})\, e^{i\vec{p}\vec{x}} + a^\dagger(\vec{p})\, e^{-i\vec{p}\vec{x}}\right), \quad (1)$$

$$\pi(\vec{x}) = i\int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{E_p}{2}} \left(a^\dagger(\vec{p})\, e^{-i\vec{p}\vec{x}} - a(\vec{p})\, e^{i\vec{p}\vec{x}}\right) \quad (2)$$

we'll derive the canonical commutation relations for the modes $a(\vec{p})$ and $a^\dagger(\vec{p})$ using the steps below.

a) Perform a Fourier transformation to deduce the relation between the Fourier modes $\tilde{\varphi}(\vec{p})$, $\tilde{\pi}(\vec{p})$ and the modes $a(\vec{p})$ and $a^\dagger(\vec{p})$.

$$\tilde{\varphi}(\vec{p}) = \int d^3x\, e^{-i\vec{p}\vec{x}} \varphi(\vec{x}) = \int d^3x\, e^{-i\vec{p}\vec{x}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}}\, e^{i\vec{q}\vec{x}} \left(a(\vec{q}) + a^\dagger(-\vec{q})\right)$$

$$= \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \left(a(\vec{q}) + a^\dagger(-\vec{q})\right) \underbrace{\int d^3x\, e^{i(\vec{p}-\vec{q})\vec{x}}}_{(2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})} = \frac{1}{\sqrt{2E_p}}\left(a(\vec{p}) + a^\dagger(-\vec{p})\right)$$

$$\tilde{\pi}(\vec{p}) = \int d^3x\, e^{-i\vec{p}\vec{x}}\, \pi(\vec{x}) = \int d^3x\, e^{-i\vec{p}\vec{x}}\, i\int \frac{d^3q}{(2\pi)^3} \sqrt{\frac{E_q}{2}}\, e^{i\vec{q}\vec{x}} \left(a^\dagger(-\vec{q}) - a(\vec{q})\right)$$

$$= i\int \frac{d^3q}{(2\pi)^3} \sqrt{\frac{E_q}{2}} \left(a^\dagger(-\vec{q}) - a(\vec{q})\right) \underbrace{\int d^3x\, e^{i(\vec{q}-\vec{p})\vec{x}}}_{(2\pi)^3 \delta^{(3)}(\vec{q}-\vec{p})} = i\sqrt{\frac{E_p}{2}}\left(a^\dagger(-\vec{p}) - a(\vec{p})\right)$$

b) Derive the commutation relation

$$[\tilde{\varphi}(\vec{p}), \tilde{\pi}(\vec{q})] = i\,(2\pi)^3\,\delta^{(3)}(\vec{p}+\vec{q}) \tag{3}$$

from the canonical commutation relations for $\varphi(\vec{x})$ and $\pi(\vec{x})$.

$$[\hat{\varphi}(\vec{p}), \hat{\pi}(\vec{q})] = \int d^3x\, e^{-i\vec{p}\vec{x}} \int d^3y\, e^{-i\vec{q}\vec{y}} \underbrace{[\varphi(\vec{x}), \pi(\vec{y})]}_{i\delta^{(3)}(\vec{x}-\vec{y})} = i\int d^3x\, e^{-i\vec{p}\vec{x}}\, e^{-i\vec{q}\vec{x}} = i(2\pi)^3\delta^{(3)}(\vec{p}+\vec{q})$$

c) Conclude that

$$[a(\vec{p}), a^\dagger(\vec{q})] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}), \qquad [a(\vec{p}), a(\vec{q})] = 0 = [a^\dagger(\vec{p}), a^\dagger(\vec{q})] \tag{4}$$

$$[a(\vec{p}), a^\dagger(\vec{q})] = \frac{1}{4}\left[\sqrt{2E_p}\,\hat{\varphi}(\vec{p}) + i\sqrt{\frac{2}{E_p}}\,\hat{\pi}(\vec{p}),\ \sqrt{2E_q}\,\hat{\varphi}(-\vec{q}) - i\sqrt{\frac{2}{E_q}}\,\hat{\pi}(-\vec{q})\right]$$

$$= \frac{1}{4}\Big(2\sqrt{E_pE_q}\,[\hat{\varphi}(\vec{p}), \hat{\varphi}(-\vec{q})] - 2i\sqrt{\frac{E_p}{E_q}}\,[\hat{\varphi}(\vec{p}), \hat{\pi}(-\vec{q})] + 2i\sqrt{\frac{E_q}{E_p}}\,[\hat{\pi}(\vec{p}), \hat{\varphi}(-\vec{q})]$$
$$+ \frac{2}{\sqrt{E_pE_q}}\,[\hat{\pi}(\vec{p}), \hat{\pi}(-\vec{q})]\Big)$$

$$= \frac{-i}{2}\left(\sqrt{\frac{E_p}{E_q}}\; i\,(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}) + \sqrt{\frac{E_q}{E_p}}\; i\,(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\right) = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})$$

$$[a(\vec{p}), a(\vec{q})] = \frac{1}{4}\left[\sqrt{2E_p}\,\hat{\varphi}(\vec{p}) + i\sqrt{\frac{2}{E_p}}\,\hat{\pi}(\vec{p}),\ \sqrt{2E_q}\,\hat{\varphi}(\vec{q}) + i\sqrt{\frac{2}{E_q}}\,\hat{\pi}(\vec{q})\right]$$

$$= \frac{1}{4}\Big(2\sqrt{E_pE_q}\,[\hat{\varphi}(\vec{p}), \hat{\varphi}(\vec{q})] + 2i\sqrt{\frac{E_p}{E_q}}\,[\hat{\varphi}(\vec{p}), \hat{\pi}(\vec{q})]$$
$$+ 2i\sqrt{\frac{E_q}{E_p}}\,[\hat{\pi}(\vec{p}), \hat{\varphi}(\vec{q})] - \frac{2}{\sqrt{E_pE_q}}\,[\hat{\pi}(\vec{p}), \hat{\pi}(\vec{q})]\Big)$$

$$= \frac{i}{2}\left(\sqrt{\frac{E_p}{E_q}}\,\delta^{(3)}(\vec{p}+\vec{q}) - \sqrt{\frac{E_q}{E_p}}\,\delta^{(3)}(\vec{p}+\vec{q})\right) = 0,$$

and similarly for $[a^\dagger(\vec{p}), a^\dagger(\vec{q})]$.

Problem 2.3 (Rotational symmetry and its corresponding generators)

Show that the action for the free classical Klein-Gordon field,

$$S = \int d^4x\, \mathcal{L} = \int d^4x \left(\frac{1}{2}\partial_\mu\phi(x)\,\partial^\mu\phi(x) - \frac{1}{2}m^2\phi^2(x)\right), \tag{8}$$

is invariant under the following three transformations

$$\delta x^j = \varepsilon\, \epsilon^{ijk}\, x_k \quad \text{with} \quad j \in \{1,2,3\}. \tag{9}$$

Furthermore, construct the corresponding Noether charges.

Since ε is a continuous parameter, we can write the transformation infinitesimally and justify an expansion in ε only to first order.

$$\phi(x) \longrightarrow \phi'(x'(x)) = \phi(x) + \varepsilon\,\delta\phi(x) + \frac{\varepsilon^2}{2}\delta^2\phi(x) + \frac{\varepsilon}{3!}\delta^3\phi(x) + \dots$$
$$= \phi(x) + \varepsilon\,\delta\phi(x) + \sigma(\varepsilon^2),$$

where $x'(x) = \Lambda x = x + \delta x$ and δx is a 4-vector given by

$$\delta x^\mu = \begin{cases} 0 & \text{for } \mu = 0 \\ \varepsilon\,\epsilon^{ij}{}_k x^k & \text{for } \mu = i \in \{1,2,3\} \text{ and } j \in \{1,2,3\} \end{cases}$$

$$j=1:\ \delta\vec{x} = \varepsilon\begin{pmatrix}0\\-x_3\\x_2\end{pmatrix},\quad j=2:\ \delta\vec{x} = \varepsilon\begin{pmatrix}x_3\\0\\-x_1\end{pmatrix},\quad j=3:\ \delta\vec{x} = \varepsilon\begin{pmatrix}-x_2\\x_1\\0\end{pmatrix}$$

$$j=1:\quad x' = x + \delta x = \begin{pmatrix}x^0\\x^1\\x^2-\varepsilon x^3\\x^3+\varepsilon x^2\end{pmatrix} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&-\varepsilon\\0&0&\varepsilon&1\end{pmatrix}x = \Lambda x$$

$$j=2:\quad x' = x + \delta x = \begin{pmatrix}x^0\\x^1+\varepsilon x^3\\x^2\\x^3-\varepsilon x^1\end{pmatrix} = \begin{pmatrix}1&0&0&0\\0&1&0&\varepsilon\\0&0&1&0\\0&-\varepsilon&0&1\end{pmatrix}x = \Lambda x$$

$$j=3:\quad x' = x + \delta x = \begin{pmatrix}x^0\\x^1-\varepsilon x^2\\x^2+\varepsilon x^1\\x^3\end{pmatrix} = \begin{pmatrix}1&0&0&0\\0&1&-\varepsilon&0\\0&\varepsilon&1&0\\0&0&0&1\end{pmatrix}x = \Lambda x$$

We can also write the transformation using the chain rule

$$\phi(x) \longrightarrow \phi'(x'(x)) = \phi(x) + \frac{\partial\phi(x)}{\partial x^\mu}\delta x^\mu + \sigma(\varepsilon^2) = \phi(x) + \partial_i\phi(x)\,\varepsilon\,\epsilon^{ij}{}_k x^k + \sigma(\varepsilon^2)$$

from which we see that $\delta\phi(x) = \partial_i\phi(x)\,\epsilon^{ij}{}_k x^k$. Therefore to first order the Lagrangian transforms as

$$\mathcal{L} \longrightarrow \mathcal{L}' = \frac{1}{2}\partial_\mu\phi'\partial^\mu\phi' - \frac{1}{2}m^2\phi'^2 = \frac{1}{2}\left[\partial_\mu(\phi+\varepsilon\delta\phi)\partial^\mu(\phi+\varepsilon\delta\phi) - m^2(\phi+\varepsilon\delta\phi)(\phi+\varepsilon\delta\phi)\right]$$

$$= \frac{1}{2}\left[\partial_\mu\phi\partial^\mu\phi + 2\partial_l\phi\,\partial^l\varepsilon\partial_i\phi\,\epsilon^{ij}{}_k x^k - m^2\phi^2 - 2m^2\phi\,\varepsilon\partial_i\phi\,\epsilon^{ij}{}_k x^k + \sigma(\varepsilon^2)\right]$$

$$= \mathcal{L} + \varepsilon\partial_l\phi\,\partial^l\partial_i\phi\,\epsilon^{ij}{}_k x^k - \varepsilon m^2\phi\,\partial_i\phi\,\epsilon^{ij}{}_k x^k + \sigma(\varepsilon^2)$$

After dropping terms of order $\mathcal{O}(\varepsilon^2)$ and above, the transf. Lagrangian reads

$$\mathcal{L}' = \mathcal{L} + \varepsilon\, \partial_l \phi\, \partial^l \partial_i \phi\, \epsilon^{ij}{}_k x^k - \varepsilon m^2 \phi \partial_i \phi\, \epsilon^{ij}{}_k x^k$$

$$= \mathcal{L} + \varepsilon \epsilon^{ij}{}_k x^k \left(\partial_l \phi\, \partial^l \partial_i \phi - m^2 \phi \partial_i \phi \right)$$

$$= \mathcal{L} + \delta x^i \left(\partial_i \left(\tfrac{1}{2} \partial_l \phi\, \partial^l \phi - \tfrac{1}{2} m^2 \phi^2 \right)\right) = \mathcal{L} + \delta x^\mu \partial_\mu \mathcal{L},$$

as one would expect from an application of the chain rule as was done for the field $\phi(x)$. Making use of the fact that ϵ^{ijk} vanishes, whenever two of its indices are identical, we can drag the x^μ in δx^μ into the derivative in front of $\mathcal{L}$, i.e.

$$\delta x^\mu \partial_\mu \mathcal{L} = \delta x^i \partial_i \mathcal{L} = \varepsilon \epsilon^{ij}{}_k x^k \partial_i \mathcal{L} = \partial_i \left(\varepsilon \epsilon^{ij}{}_k x^k \mathcal{L} \right)$$

Inserting this into the transformed action S' gives

$$S \longrightarrow S' = \int d^4x\, \mathcal{L}' = \int d^4x\, \mathcal{L} + \int d^4x\, \delta x^\mu \partial_\mu \mathcal{L} = S + \int dt \int d^3x\, \partial_i \left(\varepsilon \epsilon^{ij}{}_k x^k \mathcal{L} \right)$$

$$= S + \underbrace{\int dt \int_{\partial \mathbb{R}^3} dA_i\, \varepsilon\, \epsilon^{ij}{}_k x^k \mathcal{L}}_{0} = S,$$

where the integral over the surface of $\mathbb{R}^3$ is said to vanish because we only look at localized systems fulfilling $\mathcal{L}(\phi(x)) \xrightarrow{|x| \to \infty} 0$.
Now, we calculate the corresponding Noether charges. Firstly, the currents are given by

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta \phi - F^\mu, \quad \text{where} \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \tfrac{1}{2} \partial^\mu \phi \quad \text{and} \quad \delta \mathcal{L} = \partial_\mu F^\mu.$$

Comparing terms in our expansion of $\mathcal{L}$, we see that

$$\varepsilon\, \delta \mathcal{L} = \delta x^\mu \partial_\mu \mathcal{L} = \partial_\mu (\delta x^\mu \mathcal{L}) \quad \Longrightarrow \quad \delta \mathcal{L} = \tfrac{1}{\varepsilon} \partial_\mu (\delta x^\mu \mathcal{L})$$

$$F^\mu = \tfrac{1}{\varepsilon} \delta x^\mu \mathcal{L} \quad \Longrightarrow \quad F^0 = 0 \quad \text{because } \delta x^0 = 0$$

$$F^j = \tfrac{1}{\varepsilon}\, \varepsilon\, \epsilon^{ij}{}_k x^k \mathcal{L} = \epsilon^{ij}{}_k x^k \mathcal{L}$$

Therefore, the Noether currents take the form

$$j^0 = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \delta\phi - F^0 = \pi \delta\phi = \dot{\phi}\delta\phi$$

$$j^j = \frac{\partial \mathcal{L}}{\partial(\partial_j \phi)} \delta\phi - F^j = -\partial^j\phi\,\delta\phi - \epsilon^{ij}{}_k x^k \mathcal{L} \quad \text{for } j \in \{1,2,3\}$$

The charge is obtained via integration over j^0

$$Q = \int d^3x\, j^0 = \int d^3x\, \dot{\phi}\,\delta\phi + \int d^3x\, \dot{\phi}\, \epsilon^{ij}{}_k x_k \partial_i \phi \quad \text{for } j \in \{1,2,3\}$$

Problem 3.2 (Quantisation of the complex scalar field)

A complex scalar field $\phi(x)$ of mass m can be thought of as describing the degrees of freedom of two real scalar fields $\varphi_1(x)$ and $\varphi_2(x)$ of mass m by writing

$$\phi(x) = \frac{1}{\sqrt{2}}\left(\phi_1(x) + i\phi_2(x)\right). \qquad (2)$$

Using this relation, show the following assertions stated in the lecture:

a) The (Schrödinger picture) Hamiltonian can be written as

$$H = \int d^3x \left(\pi^\dagger(\vec{x})\pi(\vec{x}) + \vec{\nabla}\phi^\dagger(\vec{x})\,\vec{\nabla}\phi(\vec{x}) + m^2\phi^\dagger(\vec{x})\,\phi(\vec{x})\right) \qquad (3)$$

with $\pi(x) = \dot{\phi}^\dagger(x)$, $\pi^\dagger(x) = \dot{\phi}(\vec{x})$.

If the complex scalar field can be thought of as stated in the problem, it should be possible to write the Lagrangian of the complex scalar field as a sum of two real scalar field Lagrangians, i.e.

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 = \frac{1}{2}\partial_\mu\phi_1\,\partial^\mu\phi_1 - \frac{1}{2}m^2\phi_1^2 + \frac{1}{2}\partial_\mu\phi_2\,\partial^\mu\phi_2 - \frac{1}{2}m^2\phi_2^2$$

$$= \frac{1}{2}\partial_\mu\phi_1\,\partial^\mu\phi_1 + \frac{i}{2}\partial_\mu\phi_1\,\partial^\mu\phi_2 - \frac{i}{2}\partial_\mu\phi_1\,\partial^\mu\phi_2 + \frac{1}{2}\partial_\mu\phi_2\,\partial^\mu\phi_2 - \frac{1}{2}m^2(\phi_1^2 + \phi_2^2)$$

$$= \partial_\mu\frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)\,\partial^\mu\frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) - m^2\frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)\frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$$

$$= \partial_\mu\phi^\dagger\,\partial^\mu\phi - m^2\phi^\dagger\phi$$

From this Lagrangian density, we obtain the following Hamiltonian H

$$H := \int d^3x \, \mathcal{H} := \int d^3x \left(\sum_{\text{d.o.f.}} \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} \dot{\phi} - \mathcal{L} \right) = \int d^3x \left(\frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} \dot{\phi} + \frac{\partial \mathcal{L}}{\partial(\partial_t \phi^\dagger)} \dot{\phi}^\dagger - \mathcal{L} \right)$$

$$= \int d^3x \left(\dot{\phi}^\dagger \dot{\phi} + \dot{\phi} \dot{\phi}^\dagger - \partial_\mu \phi^\dagger \partial^\mu \phi + m^2 \phi^\dagger \phi \right)$$

$$= \int d^3x \left(\dot{\phi}^\dagger \dot{\phi} + \vec{\nabla} \phi^\dagger \vec{\nabla} \phi + m^2 \phi^\dagger \phi \right)$$

This expression for the Hamiltonian agrees with eq. (3), if we use the definition of the canonical conjugate momenta, i.e.

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = \dot{\phi}^\dagger \quad \text{and} \quad \pi^\dagger = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi^\dagger)} = \dot{\phi}.$$

Then we have

$$H = \int d^3x \left(\pi \pi^\dagger + \vec{\nabla} \phi^\dagger \vec{\nabla} \phi + m^2 \phi^\dagger \phi \right) = \int d^3x \left(\pi^\dagger \pi + \vec{\nabla} \phi^\dagger \vec{\nabla} \phi + m^2 \phi^\dagger \phi \right)$$

b) The mode expansion of the Schrödinger field $\phi(\vec{x})$ is

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a(\vec{p}) e^{i\vec{p}\vec{x}} + b^\dagger(\vec{p}) e^{-i\vec{p}\vec{x}} \right) \tag{4}$$

$$\phi^\dagger(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(b(\vec{p}) e^{i\vec{p}\vec{x}} + a^\dagger(\vec{p}) e^{-i\vec{p}\vec{x}} \right) \tag{5}$$

Give $a(\vec{p})$ and $b(\vec{p})$ in terms of the two fields ϕ_1 and ϕ_2.

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) = \frac{1}{\sqrt{2}} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_1(\vec{p}) e^{i\vec{p}\vec{x}} + a_1^\dagger(\vec{p}) e^{-i\vec{p}\vec{x}} \right) + \frac{i}{\sqrt{2}} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_2(\vec{p}) e^{i\vec{p}\vec{x}} + a_2^\dagger(\vec{p}) e^{-i\vec{p}\vec{x}} \right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(\frac{1}{\sqrt{2}} \left(a_1(\vec{p}) + i a_2(\vec{p}) \right) e^{i\vec{p}\vec{x}} + \frac{1}{\sqrt{2}} \left(a_1^\dagger(\vec{p}) + i a_2^\dagger(\vec{p}) \right) e^{-i\vec{p}\vec{x}} \right)$$

Comparing this result to eq (4), we see

$$a(\vec{p}) = \frac{1}{\sqrt{2}} \left(a_1(\vec{p}) + i a_2(\vec{p}) \right), \quad b^\dagger(\vec{p}) = \frac{1}{\sqrt{2}} \left(a_1^\dagger(\vec{p}) + i a_2^\dagger(\vec{p}) \right)$$

Therefore $a^\dagger(\vec{p})$ and $b(\vec{p})$ in terms of the modes of φ_1 and φ_2 are given by

$$a^\dagger(\vec{p}) = \frac{1}{\sqrt{2}} \left(a_1^\dagger(\vec{p}) - i a_2^\dagger(\vec{p}) \right), \quad b(\vec{p}) = \frac{1}{\sqrt{2}} \left(a_1(\vec{p}) - i a_2(\vec{p}) \right)$$

c) Derive the commutation relations for the modes $a(\vec{p})$, $b(\vec{p})$, $a^\dagger(\vec{p})$, and $b^\dagger(\vec{p})$. Outline how one could set up the Fock space with particles and anti-particles created from the vacuum $|0\rangle$ by $a^\dagger(\vec{p})$ and $b^\dagger(\vec{p})$.

$$[a(\vec{p}), b(\vec{q})] = \frac{1}{2}\Big([a_1(\vec{p}), a_1(\vec{q})] - i[a_1(\vec{p}), a_2(\vec{q})] + i[a_2(\vec{p}), a_1(\vec{q})] + [a_2(\vec{p}), a_2(\vec{q})]\Big) = 0$$

$$[a(\vec{p}), b^\dagger(\vec{q})] = \frac{1}{2}\Big([a_1(\vec{p}), a_1^\dagger(\vec{q})] + i[a_1(\vec{p}), a_2^\dagger(\vec{q})] + i[a_2(\vec{p}), a_1^\dagger(\vec{q})] - [a_2(\vec{p}), a_2^\dagger(\vec{q})]\Big)$$

$$= \frac{1}{2}\Big((2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q}) - (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})\Big) = 0$$

$$[a(\vec{p}), a^\dagger(\vec{q})] = \frac{1}{2}\Big([a_1(\vec{p}), a_1^\dagger(\vec{q})] - i[a_1(\vec{p}), a_2^\dagger(\vec{q})] + i[a_2(\vec{p}), a_1^\dagger(\vec{q})] + [a_2(\vec{p}), a_2^\dagger(\vec{q})]\Big)$$

$$= \frac{1}{2}\Big((2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q}) + (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})\Big) = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})$$

$$[a^\dagger(\vec{p}), b(\vec{q})] = \frac{1}{2}\Big([a_1(\vec{p}), a_1^\dagger(\vec{q})] - i[a_1(\vec{p}), a_2(\vec{q})] - i[a_2(\vec{p}), a_1(\vec{q})] - [a_2(\vec{p}), a_2^\dagger(\vec{q})]\Big)$$

$$= \frac{1}{2}\Big((2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q}) - (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})\Big) = 0$$

$$[a^\dagger(\vec{p}), b^\dagger(\vec{q})] = \frac{1}{2}\Big([a_1^\dagger(\vec{p}), a_1^\dagger(\vec{q})] + i[a_1(\vec{p}), a_2^\dagger(\vec{q})] - i[a_1(\vec{p}), a_1(\vec{q})] + [a_2^\dagger(\vec{p}), a_2^\dagger(\vec{q})]\Big) = 0$$

$$[b(\vec{p}), b^\dagger(\vec{q})] = \frac{1}{2}\Big([a_1(\vec{p}), a_1^\dagger(\vec{q})] + i[a_1(\vec{p}), a_2^\dagger(\vec{q})] - i[a_2(\vec{p}), a_1^\dagger(\vec{q})] + [a_2(\vec{p}), a_2^\dagger(\vec{q})]\Big)$$

$$= \frac{1}{2}\Big((2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q}) + (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})\Big) = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})$$

To be able to create all Fock space state's from the vacuum, we first need to establish it's existence. This can be done by looking at the 4-mom. operator

$$P^\mu = P_1^\mu + P_2^\mu = \int \frac{d^3p}{(2\pi)^3} p^\mu a_1^\dagger(\vec{p}) a_1(\vec{p}) + \int \frac{d^3p}{(2\pi)^3} p^\mu a_2^\dagger(\vec{p}) a_2(\vec{p})$$

$$= \int \frac{d^3p}{(2\pi)^3} p^\mu \frac{1}{2}\Big(a_1^\dagger(\vec{p}) a_1(\vec{p}) + a_2^\dagger(\vec{p}) a_2(\vec{p}) + a_1^\dagger(\vec{p}) a_1(\vec{p}) + a_2^\dagger(\vec{p}) a_2(\vec{p})\Big)$$

$$= \int \frac{d^3p}{(2\pi)^3} p^\mu \frac{1}{2}\Big(a_1^\dagger(\vec{p}) a_1(\vec{p}) + i a_1^\dagger(\vec{p}) a_2(\vec{p}) - i a_2^\dagger(\vec{p}) a_1(\vec{p}) + a_2^\dagger(\vec{p}) a_2(\vec{p})$$
$$+ a_1^\dagger(\vec{p}) a_1(\vec{p}) - i a_1^\dagger(\vec{p}) a_2(\vec{p}) + a_2^\dagger(\vec{p}) a_1(\vec{p}) + a_2^\dagger(\vec{p}) a_2(\vec{p})\Big)$$

$$= \int \frac{d^3p}{(2\pi)^3} p^\mu \Big(a^\dagger(\vec{p}) a(\vec{p}) + b^\dagger(\vec{p}) b(\vec{p})\Big),$$

from which we gather that the Hamiltonian given by $H = P^0$ is non-negative, i.e.

$$\langle\Psi|H|\Psi\rangle = \int \frac{d^3p}{(2\pi)^3} E_p \langle\Psi|(a^\dagger(\vec{p})a(\vec{p}) + b^\dagger(\vec{p})\, b(\vec{p}))|\Psi\rangle \geq 0 \quad \forall\, \Psi,$$

where we require that $\Psi \in \mathcal{F}$, meaning Ψ must be an element of the Fock space and therefore an Eigenstate to the number operators

$$n_+(\vec{p}) = a^\dagger(\vec{p})\, a(\vec{p}) \quad \text{and} \quad n_-(\vec{p}) = b^\dagger(\vec{p})\, b(\vec{p}) \quad \text{and the Hamiltonian!}$$

From $\langle\Psi|H|\Psi\rangle \geq 0 \;\; \forall\, \Psi \in \mathcal{F}$, we can deduce that there exists a state that fulfills

$$a(\vec{p})|0\rangle = 0 \quad \forall\, \vec{p} > \vec{0}$$

Otherwise, successive action of $a(\vec{p})$ on any state $|\Psi\rangle$ would lead to negative eigenvalues since

$$P^\mu a(\vec{p})\,|k\rangle = a(\vec{p})\, P^\mu |k\rangle - p^\mu a(\vec{p})\,|k\rangle = a(\vec{p})\, k^\mu |k\rangle - p^\mu a(\vec{p})|k\rangle = (k^\mu - p^\mu)\, a(\vec{p})|k\rangle,$$

i.e. $a(\vec{p})$ subtracts 4-momentum p^μ from any 4-momentum operator eigenstate $|k\rangle$ (where we used $[P^\mu, a(\vec{p})] = -p^\mu a(\vec{p})$). Since Fock space states with negative energies were ruled out above, we can infer the existence of such a vacuum. This vacuum we can use to set up the entire Fock space of n-particle states via employment of particle and antiparticle creation operators $a^\dagger(\vec{p})$ and $b^\dagger(\vec{p})$

$$|p_1, p_2, \dots, p_n, q_1, q_2, \dots q_r\rangle = \prod_{i=1}^{n} \sqrt{2E_{p_i}}\; a^\dagger(p_i) \prod_{j=1}^{r} \sqrt{2E_{q_j}}\; b^\dagger(\vec{q}_j)|0\rangle$$

<u>Problem 4.1</u> (Expression for $a(\vec p)$ in terms of fields)

Show that the creation and annihilation operators in a real scalar theory can be written in terms of the Heisenberg fields as

$$a^\dagger(\vec p) = -\frac{i}{\sqrt{2E_p}}\int d^3x\, e^{-ipx}\overleftrightarrow{\partial_0}\,\phi(x), \qquad a(\vec p) = \frac{i}{\sqrt{2E_p}}\int d^3x\, e^{ipx}\overleftrightarrow{\partial_0}\,\phi(x), \qquad (1)$$

where in $x := (x^0, \vec x)$ we can pick x^0 arbitrarily and $u\overleftrightarrow{\partial_0}v = u\partial_0 v - v\partial_0 u$.

Written in a mode expansion, the Heisenberg field $\phi^{(H)}(t,\vec x)$ is defined as

$$\phi^{(H)}(t,\vec x) := \phi(x) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a(\vec p)e^{-ipx} + a^\dagger(\vec p)e^{ipx}\right),$$ its time derivative is

$$\dot\phi^{(H)}(t,\vec x) := \dot\phi(x) = i\int\frac{d^3p}{(2\pi)^3}\sqrt{\frac{E_p}{2}}\left(a^\dagger(\vec p)e^{ipx} - a(\vec p)e^{-ipx}\right)$$

Plugging these expressions into the field expansions of the modes gives

$$a^\dagger(\vec q) = -\frac{i}{\sqrt{2E_q}}\int d^3x\left(e^{-iqx}\, i\int\frac{d^3p}{(2\pi)^3}\sqrt{\frac{E_p}{2}}\left(a^\dagger(\vec p)e^{ipx} - a(\vec p)e^{-ipx}\right) - \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a(\vec p)e^{-ipx} + a^\dagger(\vec p)e^{ipx}\right)(-iE_q)e^{-iqx}\right)$$

$$= \frac{1}{\sqrt{2E_q}}\int\frac{d^3p}{(2\pi)^3}\int d^3x\left(\sqrt{\frac{E_p}{2}}\left(a^\dagger(\vec p)e^{i(p-q)x} - a(\vec p)e^{-i(p+q)x}\right) + \frac{E_q}{\sqrt{2E_p}}\left(a(\vec p)e^{-i(p+q)x} + a^\dagger(\vec p)e^{i(p-q)x}\right)\right)$$

$$= \frac{1}{\sqrt{2E_q}}\left(\sqrt{\frac{E_q}{2}}\left(a^\dagger(\vec q) - a(-\vec q)\right) + \frac{E_q}{\sqrt{2E_q}}\left(a(-\vec q) + a^\dagger(\vec q)\right)\right) = \frac{1}{2}\left(a^\dagger(\vec q) - a(-\vec q) + a(-\vec q) + a^\dagger(\vec q)\right) = a^\dagger(\vec q)$$

The proove for $a(\vec q)$ follows analogously.

<u>Problem 6.2</u> (Representation of Lie algebras)

An element of a Lie group that is close to the identity can be written as

$$g(\alpha) = 1 + i\alpha^a T^a + \mathcal{O}(\alpha^2). \qquad (2)$$

The hermitian operators T^a are generators of the Lie algebra. They have the commutation relation

$$[T^a, T^b] = i f^{abc} T^c, \qquad (3)$$

with f^{abc} the structure constants. The vector space spanned by the generators with the additional Lie bracket structure in eq. (3) is called a Lie algebra.

a) Prove the identity

$$[T^a,[T^b,T^c]]+[T^b,[T^c,T^a]]+[T^c,[T^a,T^b]]=0 \tag{4}$$

and show that this implies

$$f^{ade}f^{bcd}+f^{bde}f^{cad}+f^{cde}f^{abd}=0 \tag{5}$$

$$[T^a,[T^b,T^c]]+[T^b,[T^c,T^a]]+[T^c,[T^a,T^b]]=T^aT^bT^c-T^bT^cT^a-T^aT^cT^b+T^cT^bT^a$$

$$+T^bT^cT^a-T^cT^aT^b-T^bT^aT^c+T^aT^cT^b+T^cT^aT^b-T^aT^bT^c-T^cT^bT^a+T^bT^aT^c=0$$

$$[T^a,[T^b,T^c]]+[T^b,[T^c,T^a]]+[T^c,[T^a,T^b]]=i\left(f^{bcd}[T^a,T^d]+f^{cad}[T^b,T^d]+f^{abd}[T^c,T^d]\right)$$

$$=i^2\left(f^{ade}f^{bcd}T^e+f^{bde}f^{cad}T^e+f^{cde}f^{abd}T^e\right)=-i\left(f^{ade}f^{bcd}+f^{bde}f^{cad}+f^{cde}f^{abd}\right)T^e\stackrel{!}{=}0$$

$$\Longrightarrow f^{ade}f^{bcd}+f^{bde}f^{cad}+f^{cde}f^{abd}=0$$

b) What is a representation t^a of a Lie algebra?

An n-dimensional representation R of a Group G (in this case a Lie algebra) is an automorphism, i.e. an invertible linear map,

$$R: G\longrightarrow \mathrm{End}(V),\quad R(x)=f$$

where $x\in G$ and $f: V\longrightarrow V$ is an endomorphism of some vector space V.
Further, a representation R needs to be compatible with the group action $(\cdot,\cdot)$ of G, meaning for $x,y\in V$

$$R(x)\circ R(y)=R((x,y))\quad,\quad R(x^{-1})=R^{-1}(x)$$

c) Assume that the generators in some representation r are normalized according to

$$\mathrm{tr}(t_r^a t_r^b)=C(r)\,\delta^{ab} \tag{6}$$

Show that this yields the following representation of the structure constants $f^{abc} = -\frac{i}{C(r)} \mathrm{tr}([t_r^a, t_r^b] t_r^c)$ and that f^{abc} is totally antisymmetric.

A general relation that holds true for the generators of any Lie algebra was given in eq. (3). Together with the normalization in eq. (6), this gives

$$-\frac{i}{C(r)} \mathrm{tr}([t_r^a, t_r^b] t_r^c) = -\frac{i}{C(r)} \mathrm{tr}(i f^{abd} t_r^d t_r^c) = -\frac{i^2 f^{abd}}{C(r)} \mathrm{tr}(t_r^d t_r^c) = \frac{f^{abd}}{C(r)} C(r) \delta^{dc} = f^{abc}$$

Using the invariance of the trace under cyclic permutations, we can show that f^{abc} is antisymmetric

$$f^{abc} = \underbrace{-\frac{i}{C(r)} \mathrm{tr}([t_r^a, t_r^b] t_r^c)}_{-f^{bac}} = \underbrace{-\frac{i}{C(r)} \mathrm{tr}([t_r^b, t_r^c] t_r^a)}_{f^{bca}} = -\frac{i}{C(r)} \mathrm{tr}(-[t_r^c, t_r^b] t_r^a) = -f^{cba}$$

$$= \underbrace{-\frac{i}{C(r)} \mathrm{tr}([t_r^c, t_r^a] t_r^b)}_{f^{cab}} = -\frac{i}{C(r)} \mathrm{tr}(-[t_r^a, t_r^c] t_r^b) = -f^{acb}$$

d) Similarly, show that the matrices $(t_G^a)_{dc} = i f^{abc}$ define a representation (the adjoint representation).

To prove this we make use of the Jacobi identy and the total antisymmetry of the structure constants

$$0 = f^{ade} f^{bcd} + f^{bde} f^{cad} + f^{cde} f^{abd} = (-1)^2 f^{cbd} f^{dae} - f^{cad} f^{dbe} + f^{abd} f^{cde}$$

$$= -(t_G^b)_{cd} (t_G^a)_{de} + (t_G^a)_{cd} (t_G^b)_{de} - i f^{abd} (t_G^d)_{ce}$$

$$\overset{c \leftrightarrow d}{=} (t_G^a)_{dc} (t_G^b)_{ce} - (t_G^b)_{dc} (t_G^a)_{ce} - i f^{abc} (t_G^c)_{de}$$

$$= [(t_G^a)_{dc}, (t_G^b)_{ce}] - i f^{abc} (t_G^c)_{de} \implies [(t_G^a)_{dc}, (t_G^b)_{ce}] = i f^{abc} (t_G^c)_{de}$$

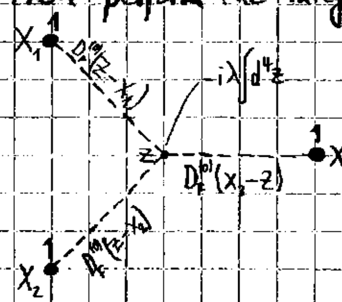
Problem 7.2 (Feynman rules in ϕ^3-theory)

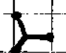
a) Give the Feynman rules for the propagator, the vertex and the external points in position space and derive from these the Feynman rules in momentum space for the $\lambda\phi^3$-theory, i.e. $\mathcal{L}_{int} = -\frac{\lambda}{3!}\phi^3$.

Feynman rules in position space for

- the propagator: $D_F^{TO}(x-y)$ - the vertex: $-i\lambda \int d^4 z_i$ - external point: 1

To obtain the corresponding Feynman rules in momentum space, we compile an expression for a vertex in $\lambda\phi^3$-theory according to the Feynman rules in position space and then perform the integral over the internal point explicitly. This procedure should not be mistaken for the Fourier transformation of the correlator. Rather it exemplifies the derivation of an equivalent set of rules to the Feynman rules in position space that rely more heavily on writing down expressions based and calculating integrals over momenta:



 $= -i\lambda\int d^4z \; 1^3 \; D_F^{(0)}(x_1-z)\, D_F^{(0)}(x_2-z)\, D_F^{(0)}(x_3-z)$

$$= -i\lambda\int d^4z \int\frac{d^4p_1}{(2\pi)^4}\frac{i e^{-ip_1(x_1-z)}}{p_1^2-m_0^2+i\varepsilon}\int\frac{d^4p_2}{(2\pi)^4}\frac{i e^{-ip_2(x_2-z)}}{p_2^2-m_0^2+i\varepsilon}\int\frac{d^4p_3}{(2\pi)^4}\frac{i e^{-ip_3(x_3-z)}}{p_3^2-m_0^2+i\varepsilon}$$

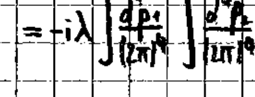
$$= -i\lambda\int\frac{d^4p_1}{(2\pi)^4}\int\frac{d^4p_2}{(2\pi)^4}\int\frac{d^4p_3}{(2\pi)^4}\frac{i}{p_1^2-m_0^2+i\varepsilon}\frac{i}{p_2^2-m_0^2+i\varepsilon}\frac{i}{p_3^2-m_0^2+i\varepsilon}e^{-ip_1x_1}e^{-ip_2x_2}e^{-ip_3x_3}\underbrace{\int d^4z\; e^{i(p_1+p_2+p_3)z}}_{(2\pi)^4\delta^{(4)}(p_1+p_2+p_3)}$$

From the above, we can deduce the Feynman rules in mom. space.

- To each line between two points, associate a factor $\frac{i}{p^2-m_0^2+i\varepsilon}$.
- To each vertex, associate a factor $-i\lambda(2\pi)^4\delta^{(4)}(\sum p_{in}-\sum p_{out})$.
- To each external point, multiply a factor of e^{-ipx} for outgoing, e^{ipx} for ingoing momenta at that point.
- Integrate over all appearing momenta $\int\frac{d^4p}{(2\pi)^4}$.

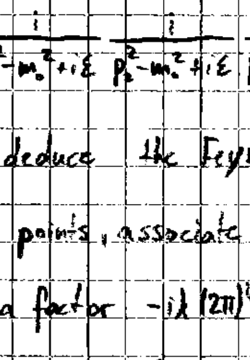
b) Calculate the symmetry factors for the following diagrams:

1)



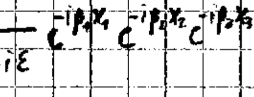
$S=2$

2)



$S=8$

3)



$S=1$

<u>Problem 9.1</u> (Spin sums and projection operator in momentum space)

The four-momentum p satisfies $p^2 = m^2$. Let

$$\Lambda_\pm = \frac{1}{2m}(m \pm \gamma\cdot p). \qquad (1)$$

Using the result $(\gamma\cdot p)^2 = m^2$, show that $\Lambda_\pm \Psi$ is an eigenspinor of $\gamma\cdot p$ with eigenvalues $\pm m$. Show further that $\Lambda_+ + \Lambda_- = 1$, $\Lambda_+\Lambda_- = \Lambda_-\Lambda_+ = 0$, and $\Lambda_\pm^2 = \Lambda_\pm$. Find vectors u_r, v_r; $r \in \{1,2\}$, such that

$$(\gamma\cdot p - m)u_r = 0, \qquad (\gamma\cdot p + m)v_r = 0, \qquad r\in\{1,2\},$$

$$\bar{u}_r u_s = -\bar{v}_r v_s = 2m\,\delta_{rs}, \qquad \bar{u}_r v_s = 0, \qquad r,s\in\{1,2\}.$$

Deduce that

$$\Lambda_+ = \sum_{r=1}^{2} \frac{1}{2m} u_r \bar{u}_r, \qquad \Lambda_- = \sum_{r=1}^{2} \frac{1}{2m} v_r \bar{v}_r. \qquad (2)$$

First we show that $\Lambda_\pm\Psi$ are Eigenspinors of the operator $\gamma\cdot p$.

$$\gamma\cdot p\,\Lambda_\pm\Psi = \gamma\cdot p\,\frac{1}{2m}(m \pm \gamma\cdot p)\Psi = \frac{1}{2m}\left(m\gamma\cdot p \pm (\gamma\cdot p)^2\right)\Psi = \frac{1}{2m}\left(m\gamma\cdot p \pm m^2\right)\Psi$$

$$= \pm m\,\frac{1}{2m}\left(m \pm \gamma\cdot p\right)\Psi = \pm m\,\Lambda_\pm\Psi$$

Now we demonstrate, that $\Lambda_\pm$ are projectors.

$$\Lambda_+ + \Lambda_- = \frac{1}{2m}(m+\gamma\cdot p) + \frac{1}{2m}(m-\gamma\cdot p) = \frac{1}{2m}(m+\gamma\cdot p + m - \gamma\cdot p) = 1$$

$$\Lambda_+\Lambda_- = \frac{1}{2m}(m+\gamma\cdot p)\frac{1}{2m}(m-\gamma\cdot p) = \frac{1}{4m^2}\left(m^2 - (\gamma\cdot p)^2\right) = 0 = \Lambda_-\Lambda_+$$

$$\Lambda_\pm^2 = \frac{1}{4m^2}(m\pm\gamma\cdot p)^2 = \frac{1}{4m^2}\left(m^2 \pm 2m\,\gamma\cdot p + (\gamma\cdot p)^2\right) = \frac{1}{4m^2}\left(2m^2 \pm 2m\,\gamma\cdot p\right) = \frac{1}{2m}(m\pm\gamma\cdot p) = \Lambda_\pm$$

We move on to find the vectors u_r, v_r, $r\in\{1,2\}$, that meet the specified requirements. Note therefore that the product $\gamma\cdot p$ can be written as

$$\gamma\cdot p = \gamma_\mu p^\mu = \begin{pmatrix} 0 & \sigma_\mu p^\mu \\ \bar{\sigma}^\mu p^\mu & 0 \end{pmatrix}, \text{ where } \sigma_0 = \mathbb{1}_2 \text{ and } \sigma_i,\ i\in\{1,2,3\} \text{ the Pauli matrices}$$

$$(\gamma \cdot p - m)\, u_r = \begin{pmatrix} -m\mathbb{1}_2 & \sigma_\mu p^\mu \\ \sigma^\mu p^\mu & -m\mathbb{1}_2 \end{pmatrix}\begin{pmatrix} u_{r,1} \\ u_{r,2} \end{pmatrix} = \begin{pmatrix} -m\,u_{r,1} + \sigma_\mu p^\mu u_{r,2} \\ \sigma^\mu p^\mu u_{r,1} - m\,u_{r,2} \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Longrightarrow \quad u_{r,1} = \frac{1}{m}\sigma_\mu p^\mu u_{r,2}\,, \quad u_{r,2} = \frac{1}{m}\sigma^\mu p^\mu u_{r,1} = \frac{1}{m^2}\sigma^\mu p^\mu \sigma_\nu p^\nu u_{r,2} = \frac{p^2}{m^2} u_{r,2} \ \checkmark$$

where we assumed that $u_{r,1}$ is a two-column vector (or 2×2-matrix) to make the contraction $\mathbb{1}_2 u_{r,i} = u_{r,i}$. The same will be done for v_r.

$$(\gamma \cdot p - m)\, v_r = \begin{pmatrix} m\mathbb{1}_2 & \sigma_\mu p^\mu \\ \sigma^\mu p^\mu & m\mathbb{1}_2 \end{pmatrix}\begin{pmatrix} v_{r,1} \\ v_{r,2} \end{pmatrix} = \begin{pmatrix} m\,v_{r,1} + \sigma_\mu p^\mu v_{r,2} \\ \sigma^\mu p^\mu v_{r,1} + m\,v_{r,2} \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Longrightarrow \quad v_{r,1} = -\frac{1}{m}\sigma_\mu p^\mu v_{r,2}\,, \quad v_{r,2} = -\frac{1}{m}\sigma^\mu p^\mu v_{r,1} = \frac{(-1)^2}{m^2}\sigma^\mu p^\mu \sigma_\nu p^\nu v_{r,2} = \frac{p^2}{m^2} v_{r,2} \ \checkmark$$

To show that both equations for u_r and v_r are consistent, we used

$$\sigma^\mu p^\mu \sigma_\nu p^\nu = \sigma^0 p^0 \sigma^0 p^0 - \sigma^i p^i \sigma^j p^j = (p^0 \mathbb{1}_2)^2 - p^i p^j \sigma^i \sigma^j = (p^0)^2 \mathbb{1}_2 - p^i p^j \delta_{ij} \mathbb{1}_2$$
$$= (p^0)^2 \mathbb{1}_2 - \vec{p}^{\,2} \mathbb{1}_2 = p^2 \mathbb{1}_2 = m^2 \mathbb{1}_2$$

So far, we found u_r and v_r to be

$$u_r = \begin{pmatrix} u_{r,1} \\ u_{r,2} \end{pmatrix} = \begin{pmatrix} \frac{1}{m}\sigma_\mu p^\mu u_{r,2} \\ u_{r,2} \end{pmatrix} \qquad v_r = \begin{pmatrix} v_{r,1} \\ v_{r,2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{m}\sigma_\mu p^\mu v_{r,2} \\ v_{r,2} \end{pmatrix}$$

One can then make the conventional choice to set $u_{r,2} = \sqrt{p_\mu \sigma_\mu}\, \xi_r = v_{r,2}$

$$\frac{1}{m}\sigma_\mu p^\mu u_{r,2} = \frac{1}{m}\sqrt{\sigma_\mu p^\mu}\sqrt{\sigma_\mu p^\mu}\sqrt{p_\mu \sigma_\mu}\,\xi_r = \frac{1}{m}\sqrt{\sigma_\mu p^\mu}\sqrt{\sigma_\mu p^\mu \underbrace{p_\nu \sigma_\nu}_{m^2}}\,\xi_r = \sqrt{\sigma_\mu p^\mu}\,\xi_r$$

Therefore

$$u_r = \begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\,\xi_r \\ \sqrt{\sigma_\mu p_\mu}\,\xi_r \end{pmatrix}, \qquad v_r = \begin{pmatrix} -\sqrt{\sigma_\mu p^\mu}\,\xi_r \\ \sqrt{\sigma_\mu p_\mu}\,\xi_r \end{pmatrix}$$

We now introduce a basis of the space of two spinors by ξ_s with

$$\xi_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \xi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \text{such that } \xi_r^\dagger \xi_s = \delta_{rs}$$

We then have

$$\bar{u}_r u_s = u_r^\dagger \gamma^0 u_s = u_{r,1}^\dagger u_{s,2} + u_{r,2}^\dagger u_{s,1} = \sqrt{\sigma_\mu p^\mu \sigma_\nu p_\nu}\,\xi_r^\dagger \xi_s + \sqrt{\sigma_\mu p_\mu \sigma_\nu p^\nu}\,\xi_r^\dagger \xi_s$$
$$= 2m\,\delta_{rs} = \bar{v}_r v_s$$

$$\sum_{r=1}^{2} u_r \bar{u}_r = \sum_{r=1}^{2} u_r \gamma^0 u_r^\dagger = \sum_{r=1}^{2} \begin{pmatrix} u_{r,1} \\ u_{r,2} \end{pmatrix} \left(u_{r,2}^\dagger, u_{r,1}^\dagger \right) = \sum_{r=1}^{2} \begin{pmatrix} u_{r,1} u_{r,2}^\dagger & u_{r,1} u_{r,1}^\dagger \\ u_{r,2} u_{r,2}^\dagger & u_{r,2} u_{r,1}^\dagger \end{pmatrix}$$

$$= \sum_{r=1}^{2} \begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\, \xi_r \sqrt{\bar\sigma_\mu p^\mu}^{\,\dagger} \xi_r^\dagger & \sqrt{\sigma_\mu p^\mu}\, \xi_r \sqrt{\sigma_\mu p^\mu}\, \xi_r^\dagger \\ \sqrt{\bar\sigma_\mu p^\mu}\, \xi_r \sqrt{\bar\sigma_\mu p^\mu}^{\,\dagger} \xi_r^\dagger & \sqrt{\bar\sigma_\mu p^\mu}\, \xi_r \sqrt{\sigma_\mu p^\mu}\, \xi_r^\dagger \end{pmatrix}$$

$$= \sum_{r=1}^{2} \begin{pmatrix} \sqrt{\sigma_\mu p^\mu \bar\sigma_\nu p^\nu}\, \xi_r \xi_r^\dagger & \sigma_\mu p^\mu \xi_r \xi_r^\dagger \\ \sqrt{\bar\sigma_\mu p^\mu}^{\,2} \xi_r \xi_r^\dagger & \sqrt{\bar\sigma_\mu p^\mu \sigma_\nu p^\nu}\, \xi_r \xi_r^\dagger \end{pmatrix} = \begin{pmatrix} m \mathbb{1}_2 & \sigma_\mu p^\mu \\ \bar\sigma_\mu p^\mu & m \mathbb{1}_2 \end{pmatrix} = \gamma p + m$$

The same holds for $\sum_{r=1}^{2} v_r \bar{v}_r$ except for an overall minus sign. Therefore

$$\Lambda_+ = \sum_{r=1}^{2} \frac{1}{2m} u_r \bar{u}_r = \frac{1}{2m} (\gamma p + m) \qquad \Lambda_- = -\sum_{r=1}^{2} \frac{1}{2m} v_r \bar{v}_r = \frac{1}{2m} (\gamma p - m)$$

<u>Problem 9.2</u> (Gamma matrices and SO(1,3))

The Clifford algebra for γ-matrices is described by the anticommutation relations $\{\gamma^\mu, \gamma^\nu\} = 2\eta_{\mu\nu} \mathbb{1}_4$. (3)

a) Show that

$$[\gamma^\kappa \gamma^\lambda, \gamma^\mu \gamma^\nu] = 2\eta^{\lambda\mu} \gamma^\kappa \gamma^\nu - 2\eta^{\kappa\mu} \gamma^\lambda \gamma^\nu + 2\eta^{\lambda\nu} \gamma^\mu \gamma^\kappa - 2\eta^{\kappa\nu} \gamma^\mu \gamma^\lambda.$$

$$[\gamma^\kappa \gamma^\lambda, \gamma^\mu \gamma^\nu] = [\gamma^\kappa \gamma^\lambda, \gamma^\mu] \gamma^\nu + \gamma^\mu [\gamma^\kappa \gamma^\lambda, \gamma^\nu] = \gamma^\kappa [\gamma^\lambda, \gamma^\mu] \gamma^\nu + [\gamma^\kappa, \gamma^\mu] \gamma^\lambda \gamma^\nu$$

$$+ \gamma^\mu \gamma^\kappa [\gamma^\lambda, \gamma^\nu] + \gamma^\mu [\gamma^\kappa, \gamma^\nu] \gamma^\lambda$$

$$= \gamma^\kappa \gamma^\lambda \gamma^\mu \gamma^\nu - \gamma^\kappa \gamma^\mu \gamma^\lambda \gamma^\nu + \gamma^\kappa \gamma^\mu \gamma^\lambda \gamma^\nu - \gamma^\mu \gamma^\kappa \gamma^\lambda \gamma^\nu$$

$$+ \gamma^\mu \gamma^\kappa \gamma^\lambda \gamma^\nu - \gamma^\mu \gamma^\kappa \gamma^\nu \gamma^\lambda + \gamma^\mu \gamma^\kappa \gamma^\nu \gamma^\lambda - \gamma^\mu \gamma^\nu \gamma^\kappa \gamma^\lambda$$

$$= \gamma^\kappa \{\gamma^\lambda, \gamma^\mu\} \gamma^\nu - \{\gamma^\kappa, \gamma^\mu\} \gamma^\lambda \gamma^\nu + \gamma^\mu \gamma^\kappa \{\gamma^\lambda, \gamma^\nu\} - \gamma^\mu \{\gamma^\kappa, \gamma^\nu\} \gamma^\lambda$$

$$= \gamma^\kappa 2\eta^{\lambda\mu} \mathbb{1}_4 \gamma^\nu - 2\eta^{\kappa\mu} \gamma^\lambda \gamma^\nu + \gamma^\mu \gamma^\kappa 2\eta^{\lambda\nu} \mathbb{1}_4 - \gamma^\mu 2\eta^{\kappa\nu} \mathbb{1}_4 \gamma^\lambda$$

$$= 2\eta^{\lambda\mu} \gamma^\kappa \gamma^\nu - 2\eta^{\kappa\mu} \gamma^\lambda \gamma^\nu + 2\eta^{\lambda\nu} \gamma^\mu \gamma^\kappa - 2\eta^{\kappa\nu} \gamma^\mu \gamma^\lambda$$

b) By expressing $S^{\kappa\lambda} = \frac{1}{4}[\gamma^\kappa, \gamma^\lambda]$ as $\frac{1}{2}(\gamma^\kappa \gamma^\lambda - \eta^{\kappa\lambda})$, etc., evaluate $[S^{\kappa\lambda}, S^{\mu\nu}]$.

$$S = \frac{1}{4}[\gamma^\kappa, \gamma^\lambda] = \frac{1}{4}(\gamma^\kappa \gamma^\lambda - \gamma^\lambda \gamma^\kappa) = \frac{1}{4}(\gamma^\kappa \gamma^\lambda - (-\gamma^\kappa \gamma^\lambda + 2\eta^{\kappa\lambda} \mathbb{1}_4))$$

$$= \frac{1}{2}(\gamma^\kappa \gamma^\lambda - \eta^{\kappa\lambda} \mathbb{1}_4)$$

$$[S^{\kappa\lambda}, S^{\mu\nu}] = \frac{1}{4}\left[\gamma^\kappa\gamma^\lambda - \eta^{\kappa\lambda}\mathbb{1}_4, \gamma^\mu\gamma^\nu - \eta^{\mu\nu}\mathbb{1}_4\right] = \frac{1}{4}\Big([\gamma^\kappa\gamma^\lambda, \gamma^\mu\gamma^\nu] - \underbrace{[\gamma^\kappa\gamma^\lambda, \eta^{\mu\nu}\mathbb{1}_4]}_{0}$$

$$- \underbrace{[\eta^{\kappa\lambda}\mathbb{1}_4, \gamma^\mu\gamma^\nu]}_{0} + \underbrace{[\eta^{\kappa\lambda}\mathbb{1}_4, \eta^{\mu\nu}\mathbb{1}_4]}_{0}\Big)$$

$$= \frac{1}{4}\left(2\eta^{\lambda\mu}\gamma^\kappa\gamma^\nu - 2\eta^{\kappa\mu}\gamma^\lambda\gamma^\nu + 2\eta^{\lambda\nu}\gamma^\mu\gamma^\kappa - 2\eta^{\kappa\nu}\gamma^\mu\gamma^\lambda\right)$$

$$= \frac{1}{2}\Big(\eta^{\lambda\mu}\gamma^\kappa\gamma^\nu - \eta^{\lambda\mu}\eta^{\kappa\nu} + \underline{\eta^{\lambda\mu}\eta^{\kappa\nu}} - \eta^{\kappa\mu}\gamma^\lambda\gamma^\nu + \eta^{\kappa\mu}\eta^{\lambda\nu} - \underline{\eta^{\kappa\mu}\eta^{\lambda\nu}}$$

$$+ \eta^{\lambda\nu}\gamma^\mu\gamma^\kappa - \eta^{\lambda\nu}\eta^{\mu\kappa} + \underline{\eta^{\lambda\nu}\eta^{\mu\kappa}} - \eta^{\kappa\nu}\gamma^\mu\gamma^\lambda + \eta^{\kappa\nu}\eta^{\mu\lambda} - \underline{\eta^{\kappa\nu}\eta^{\mu\lambda}}\Big)$$

$$= \eta^{\lambda\mu}S^{\kappa\nu} - \eta^{\kappa\mu}S^{\lambda\nu} + \eta^{\lambda\nu}S^{\mu\kappa} - \eta^{\kappa\nu}S^{\mu\lambda}$$

c) Show that, if $S^i = \frac{i}{4}\varepsilon_{ijk}\gamma^j\gamma^k$, then $[S^i, S^j] = i\varepsilon_{ijk}S^k$.

$$S^i = \frac{i}{4}\varepsilon_{ijk}\gamma^j\gamma^k = \frac{i}{8}\varepsilon_{ijk}\left(\gamma^j\gamma^k - \gamma^k\gamma^j\right) = \frac{i}{8}\varepsilon_{ijk}[\gamma^j, \gamma^k] = \frac{1}{2}\varepsilon_{ijk}S^{jk}$$

$$\Longrightarrow \quad i\varepsilon_{ijk}S^k = i\varepsilon_{ijk}\frac{1}{2}\varepsilon_{klm}S^{lm} = \frac{1}{2}\varepsilon_{kij}\varepsilon_{klm}S^{lm} = \frac{1}{2}\left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)S^{lm}$$

$$= \frac{1}{2}\left(\delta_{il}\delta_{jm}S^{lm} - \delta_{im}\delta_{jl}S^{lm}\right) = \frac{1}{2}\left(S^{ij} - S^{ji}\right) = \frac{1}{2}\left(S^{ij} + S^{ij}\right) = S^{ij}$$

$$[S^i, S^j] = -\frac{1}{4}\varepsilon_{ikl}\varepsilon_{jmn}[S^{kl}, S^{mn}] = -\frac{1}{4}\varepsilon_{ikl}\varepsilon_{jmn}\left(-\delta_{lm}S^{kn} + \delta_{km}S^{ln} - \delta_{ln}S^{mk} + \delta_{kn}S^{ml}\right)$$

In the above expression, the indices $k, l, m,$ and n are summed over. Hence, they can be freely relabelled.

$$= \frac{1}{4}\varepsilon_{ikl}\varepsilon_{jmn}\left(\delta_{lm}S^{kn} + \delta_{km}S^{nl} + \delta_{ln}S^{mk} + \delta_{kn}S^{ml}\right)$$

$$= \varepsilon_{ikl}\varepsilon_{jmn}\delta_{lm}S^{kn} = \varepsilon_{ikm}\varepsilon_{jmn}S^{kn} = \varepsilon_{mik}\varepsilon_{mnj}S^{kn}$$

$$= \left(\delta_{kj}\delta_{in} - \delta_{kn}\delta_{ij}\right)S^{kn} = S^{ij} - \delta_{ij}S^{nn} = S^{ij} = i\varepsilon_{ijk}S^k$$

d) Show also that $[\gamma^0, S^i] = [\gamma^5, S^i] = 0$, where $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.

$$[\gamma^0, S^i] = \frac{i}{4}\varepsilon_{ijk}[\gamma^0, \gamma^j\gamma^k] = 0$$

$[\gamma^5, S^i] = \frac{i^2}{4}\varepsilon_{ijk}\left(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^j\gamma^k - \gamma^j\gamma^k\gamma^0\gamma^1\gamma^2\gamma^3\right) = \frac{i^2}{4}\varepsilon_{ijk}\left(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^j\gamma^k - (-1)\right.$[illegible]$\left.\gamma^0\gamma^1\gamma^2\gamma^3\gamma^j\gamma^k\right) = 0$

e) Now show that $(S^1)^2 = (S^2)^2 = (S^3)^2 = \frac{1}{4}$.

$$(S^1)^2 = \left(\frac{i}{4}\varepsilon_{1jk}\gamma^j\gamma^k\right)^2 = -\frac{1}{16}\left(\gamma^2\gamma^3 - \gamma^3\gamma^2\right)^2 = -\frac{1}{16}\Big(2\cdot\underbrace{\tfrac{1}{2}[\gamma^2,\gamma^3]}_{\gamma^2\gamma^3-\eta^{23}}\Big)^2 = -\frac{1}{4}\left(\gamma^2\gamma^3\right)^2$$

$$= -\frac{1}{4}\gamma^2\gamma^3\gamma^2\gamma^3 = -\frac{1}{4}\gamma^2\Big(-\gamma^2\gamma^3 + \underbrace{2\eta^{32}}_{0}\Big)\gamma^3 = \frac{1}{4}\gamma^2\gamma^2\gamma^3\gamma^3 = \frac{1}{4}(-\mathbb{1}_4)(-\mathbb{1}_4) = \frac{1}{4}\mathbb{1}_4$$

$(S^2)^2 = (S^3)^2 = \frac{1}{4}\mathbb{1}_4$ can be shown analogously.

f) Verify these results for the particular representation

$$\gamma^0 = \begin{pmatrix}\mathbb{1}_2 & 0\\ 0 & -\mathbb{1}_2\end{pmatrix}, \qquad \gamma^i = \begin{pmatrix}0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix} \quad \text{and show} \quad S^i = \frac{1}{2}\begin{pmatrix}\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix}, \quad \gamma^5 = \begin{pmatrix}0 & \mathbb{1}_2\\ \mathbb{1}_2 & 0\end{pmatrix}$$

in this representation. What can you deduce about rotations and spin in the Dirac field theory.

To show the explicit form of S^i, we first take a look at the Pauli matrices. They both powerful commutation and anticommutation relations.

$$[\sigma^i, \sigma^j] = 2i\,\varepsilon_{ijk}\sigma^k, \qquad \{\sigma^i, \sigma^j\} = \sigma^i\sigma^j + \sigma^j\sigma^i = 2\delta^{ij}\mathbb{1}_2$$

From adding a term $\sigma^i\sigma^j$ to the the anticommutator, we see that

$$2\sigma^i\sigma^j = \sigma^i\sigma^j - \sigma^j\sigma^i + 2\delta^{ij}\mathbb{1}_2 = 2i\,\varepsilon_{ijk}\sigma^k + 2\delta^{ij}\mathbb{1}_2$$

$$\varepsilon_{ijk}\sigma^k\sigma^j = i\underbrace{\varepsilon_{ijk}\varepsilon_{jkl}}_{2\delta_{il}}\sigma^l + \underbrace{\varepsilon_{ijk}\delta^{jk}}_{0}\mathbb{1}_2 = 2i\sigma^i$$

With the above relation, it is easy.

$$S^i = \frac{1}{4}\varepsilon_{ijk}\gamma^j\gamma^k = \frac{i}{4}\varepsilon_{ijk}\begin{pmatrix}0 & \sigma_j\\ -\sigma_j & 0\end{pmatrix}\begin{pmatrix}0 & \sigma_k\\ -\sigma_k & 0\end{pmatrix} = \frac{i}{4}\varepsilon_{ijk}\begin{pmatrix}-\sigma_j\sigma_k & 0\\ 0 & -\sigma_j\sigma_k\end{pmatrix} = \frac{i}{4}(-2i)\begin{pmatrix}\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix}$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = i\gamma^0\gamma^1\begin{pmatrix}-\sigma_2\sigma_3 & 0\\ 0 & -\sigma_2\sigma_3\end{pmatrix} = i\gamma^0\begin{pmatrix}0 & -\sigma_1\sigma_2\sigma_3\\ \sigma_1\sigma_2\sigma_3 & 0\end{pmatrix} = i\begin{pmatrix}0 & -\sigma_1\sigma_2\sigma_3\\ -\sigma_1\sigma_2\sigma_3 & 0\end{pmatrix} = \begin{pmatrix}0 & \mathbb{1}_2\\ \mathbb{1}_2 & 0\end{pmatrix},$$

where we used in the last step that the gamma matrices are involuntary, i.e. $-i\sigma_1\sigma_2\sigma_3 = \mathbb{1}_2$. Since $S^{\gamma\lambda}$ satisfies the Lorentz algebra, we showed that the S^i are a subalgebra of $so(1,3)$. The explicit form of S^i is block diagonal, where each block alone satisfies the $su(2)$ algebra. Since the blocks are 2-dim, S^i form a $\frac{1}{2}\oplus\frac{1}{2}$ representation.

Problem 9.4 (Global symmetries of the free Dirac action)

a) Under global translations $x \to x + \epsilon$ the Dirac spinor field transforms as

$$\Psi(x) \to \Psi(x) - \epsilon^\mu \partial_\mu \Psi(x) + \sigma(\epsilon^2). \tag{7}$$

Review the discussion of the canonical energy momentum tensor in the free scalar theory and deduce the form of the canonical energy momentum tensor $T^{\mu\nu}$ for the free Dirac action. Also give the form of the conserved 4-momentum operator P^μ and compare P^0 with the canonical Hamiltonian.

Hint: Since the energy-momentum is only conserved on-shell, we can implement the equations of motion directly in $T^{\mu\nu}$. The final result will then be

$$T^{\mu\nu} = i\,\overline{\Psi}(x)\,\gamma^\mu \partial^\nu \Psi(x). \tag{8}$$

From eq. (7) and $\Psi(x) \to \Psi(x) + \epsilon^\mu \delta_\mu \Psi(x) + \sigma(\epsilon^2)$, we gather that

$$\delta_\mu \Psi(x) = -\partial_\mu \Psi(x) = -\frac{\partial}{\partial x^\mu} \Psi(x).$$

$\mathcal{L}$ is known to be a local function of x and therefore transforms as

$$\mathcal{L} \to \mathcal{L} - \epsilon^\nu \partial_\nu \mathcal{L} + \sigma(\epsilon^2)$$

Again, from comparing the above to an expansion in orders of variations,

$$\mathcal{L} \to \mathcal{L} + \epsilon^\nu \delta_\nu \mathcal{L} + \sigma(\epsilon^2),$$

we get that $\delta_\nu \mathcal{L} = -\partial_\nu \mathcal{L} = -\partial_\mu \eta^\mu_{\ \nu} \mathcal{L}$. $(F^\mu)_\nu$ as in $\delta_\nu \mathcal{L} = \partial_\mu (F^\mu)_\nu$ is given by $(F^\mu)_\nu = -\eta^\mu_\nu \mathcal{L}$. Even though the first variation of the second degree of freedom of the Dirac field $\delta\Psi^+(x)$ appears in the

Noether current as well, it does not need to be computed here because its prefactor $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^\dagger)} = 0$ vanishes. The four Noether currents $(j^\mu)_\nu$ (one for each continuous symmetry parameter in ε^ν) can be written as

$$(j^\mu)_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi)} \delta_\nu \Psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^\dagger)} \delta_\nu \Psi^\dagger - F^\mu = \bar{\Psi} i \gamma^\mu (-\partial_\nu \Psi) + \eta^\mu_\nu \mathcal{L}$$

Therefore, the energy-momentum-tensor $T^{\mu\nu} = (j^\mu)^\nu$ on-shell, i.e. using $(i\gamma^\mu \partial_\mu - m)\Psi = 0$ in the Lagrangian, reads

$$T^{\mu\nu} = -i \bar{\Psi} \gamma^\mu \partial_\nu \Psi$$

The conserved 4-momentum operator $P^\mu := T^{\mu 0}$ and the Hamiltonian $H := T^{00}$ are

$$P^\mu = -i \bar{\Psi} \gamma^\mu \partial_t \Psi \quad , \quad H = -i \bar{\Psi} \gamma^0 \partial_t \Psi = -i \Psi^\dagger \partial_t \Psi$$

b) Give the infinitesimal version of the transformation of the Dirac spinor under a Lorentz transformation

$$x \longrightarrow \exp\left(-\frac{i}{2} \omega_{\mu\nu} \mathcal{J}^{\mu\nu}\right) x \tag{9}$$

with $\mathcal{J}^{\mu\nu}$ as given in the lecture. Deduce that the conserved current associated with Lorentz invariance can be written as

$$(\mathcal{J}^\mu)^{\rho\sigma} = x^\sigma T^{\mu\rho} - x^\rho T^{\mu\sigma} - \bar{\Psi} \gamma^\mu S^{\rho\sigma} \Psi. \tag{10}$$

$\partial_\mu \mathcal{J}^\mu$ vanishes on-shell.

To find the spinor representation of the Lorentz algebra $so(1,3)$ we start from the Clifford algebra $Cliff(1,3)$ defined as the algebra spanned by $n \times n$-matrices $(\gamma^\mu)^A{}_B$, $\mu \in \{0,1,2,3\}$ and $A, B \in \{1, \dots, n\}$ such that the following anticommutator is fulfilled

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \mathbb{1}_4 .$$

The central point is that given matrices $(\gamma^\mu)^A{}_B$ as above, the objects $(S^{\rho\sigma})^A{}_B$ defined as

$$(S^{\rho\sigma})^A{}_B := \frac{i}{4}[\gamma^\rho, \gamma^\sigma]^A{}_B$$

form a representation of $so(1,3)$, meaning they obey the Lorentz algebras defining relation $\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta^{\rho\sigma} = \eta^{\mu\nu}$ when expanded (to second order)

$$[S^{\rho\sigma}, S^{\tau\kappa}] = -i\left(\eta^{\rho\kappa} S^{\sigma\tau} + \eta^{\sigma\tau} S^{\rho\kappa} - \eta^{\rho\tau} S^{\sigma\kappa} - \eta^{\sigma\kappa} S^{\rho\tau}\right)$$

Thus, every representation of $Cliff(1,3)$ induces a representation of $so(1,3)$. Now, as stated in the problem, a finite Lorentz transformation $\Lambda^\mu{}_\nu$ is given by

$$\Lambda^\mu{}_\nu = \left[e^{-\frac{i}{2}\omega_{\rho\sigma}\mathcal{J}^{\rho\sigma}}\right]^\mu{}_\nu, \quad \text{where } (\mathcal{J}^{\rho\sigma})^{\mu\nu} := i\left(\eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\rho\nu}\eta^{\sigma\mu}\right).$$

A spinor representation for every one of these $\Lambda \in SO(1,3)$ is given by

$$[S(\Lambda)]^A{}_B = \left[e^{-\frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}}\right]^A{}_B,$$

where $A, B \in \{1,2,3,4\}$ for $Cliff(1,3)$ are called spinor indices. We are to an infintesimal version of the Dirac spinor transformation under the above Lorentz transformation. In general (also finitely), a Dirac spinor transforms as

$$\psi^A(x) \longrightarrow [S(\Lambda)]^A{}_B \, \psi^B(\Lambda^{-1}x').$$

The find the infinetsimal version, we insert here the above expressions for $S(\Lambda)$ and Λ and expand to first order in $\omega_{\rho\sigma}$.

$$(\Lambda^{-1})^\mu{}_\nu = \left[\mathbb{1}_4 + \frac{i}{2}\omega_{\rho\sigma}\mathcal{J}^{\rho\sigma} + \mathcal{O}(\omega_{\rho\sigma}^2)\right]^\mu{}_\nu \approx \delta^\mu_\nu - \frac{1}{2}\omega_{\rho\sigma}\left(\eta^{\rho\mu}\eta^\sigma_\nu - \eta^\rho_\nu\eta^{\sigma\mu}\right)$$

$$= \delta^\mu_\nu - \frac{1}{2}\left(\omega^\mu{}_\nu - \omega_\nu{}^\mu\right) = \delta^\mu{}_\nu - \frac{1}{2}\left(\omega^\mu{}_\nu - \eta^\alpha_\nu \omega^{\nu\mu}\right)$$

$$= \delta^\mu{}_\nu - \frac{1}{2}\left(\omega^\mu{}_\nu + \eta^\alpha_\nu \omega^{\mu\nu}\right) = \delta^\mu{}_\nu - \frac{1}{2}\left(\omega^\mu{}_\nu + \omega^\mu{}_\nu\right) = \delta^\mu{}_\nu - \omega^\mu{}_\nu$$

$$[S(\Lambda)]^A{}_B = \left[\mathbb{1}_4 - \frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma} + \mathcal{O}(\omega_{\rho\sigma}^2)\right]^A{}_B \approx \delta^A{}_B + \frac{1}{8}\omega_{\rho\sigma}[\gamma^\rho, \gamma^\sigma]^A{}_B$$

$$= \delta^A{}_B + \frac{1}{8}\Big[\omega_{\rho\sigma}\gamma^\rho\gamma^\sigma + \underbrace{\omega_{\rho\sigma}\gamma^\sigma\gamma^\rho}_{\rho\longleftrightarrow\sigma}\Big]^A{}_B = \delta^A{}_B + \frac{1}{4}\left[\omega_{\rho\sigma}\gamma^\rho\gamma^\sigma\right]^A{}_B$$

Therefore, to first order a Dirac spinor transforms as

$$\Psi(x) \longrightarrow [S(\Lambda)]^A{}_B\Psi^B(\Lambda^{-1}x') = \left(\delta^A{}_B + \frac{1}{4}[\omega_{\rho\sigma}\gamma^\rho\gamma^\sigma]^A{}_B\right)\Psi^B((\delta^\mu{}_\nu - \omega^\mu{}_\nu)x'^\nu)$$

$$= \Psi^A(x'^\mu - \underbrace{\omega^\mu{}_\nu x'^\nu}_{\varepsilon^\mu}) + \frac{1}{4}[\omega_{\rho\sigma}\gamma^\rho\gamma^\sigma]^A{}_B\Psi^B(x'^\mu - \omega^\mu{}_\nu x'^\nu)$$

$$\approx \Psi^A(x'^\mu) - \omega^\mu{}_\nu x'^\nu\partial_\mu\Psi^A(x'^\mu) + \frac{1}{4}[\omega_{\rho\sigma}\gamma^\rho\gamma^\sigma]^A{}_B\Psi^B(x'^\mu) + \mathcal{O}[(\omega^\mu{}_\nu x'^\nu)^2]$$

The conserved current associated with this symmetry transformation will not be derived here.

c) Show that the Dirac action exhibits the following global $U(1)$ symmetry

$$\Psi(x) \rightarrow e^{i\alpha}\Psi(x), \qquad \overline{\Psi}(x) \longrightarrow \overline{\Psi}(x)\, e^{-i\alpha}, \qquad \alpha \in \mathbb{R}$$

and deduce the corresponding Noether current $j^\mu = -\overline{\Psi}\gamma^\mu\Psi$. Also give the form of the Noether charge $\hat{Q}$ in terms of the modes of $\Psi(x)$.

The Dirac action under a global $U(1)$ transformation behaves as follows

$$S' = \int d^4x\, \mathcal{L}' = \int d^4x\, \overline{\Psi}'(i\gamma^\mu\partial_\mu - m)\Psi' = \int d^4x\, \overline{\Psi}e^{-i\alpha}(i\gamma^\mu\partial_\mu - m)e^{i\alpha}\Psi$$

$$= \int d^4x\, \overline{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi = \int d^4x\, \mathcal{L} = S$$

It remains unchanged, hence the above $U(1)$ transformation is really a symmetry. Since the transformation is generated by a continuous parameter $\alpha \in \mathbb{R}$, we can write it infinitesimally by expanding $e^{i\alpha}$ to first order in α

$$\Psi(x) \longrightarrow \Psi'(x) = e^{i\alpha}\Psi(x) = \Psi(x) + i\alpha\Psi(x) + \mathcal{O}(\alpha^2)$$

By comparing this expression with an expansion of $\Psi'(x)$ in orders of variations, i.e. $\Psi'(x) = \Psi(x) + \alpha\,\delta\Psi(x) + \mathcal{O}(\alpha^2)$, we see that $\delta\Psi = i\Psi$. From $\mathcal{L}' = \mathcal{L}$, we know $\delta\mathcal{L} = 0$ and therefore F^μ is constant ($\delta\mathcal{L} = \partial_\mu F^\mu$) and can be set to zero as constant terms in a current are of no physical interest. j^μ then reads

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)}\delta\Psi + \underbrace{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi^+)}}_{0}\delta\Psi^+ - F^\mu = \overline{\Psi} i\gamma^\mu i\Psi = -\overline{\Psi}\gamma^\mu\Psi$$

The Noether charge Q in terms of the modes of the field $\Psi(x)$ can be obtained by integrating the current's zero component over space and inserting the free mode expansions for $\Psi(x)$ and $\Psi^+(x)$.

$$Q = \int d^3x\, j^0 = \int d^3x\,(-\overline{\Psi}\gamma^0\Psi) = -\int d^3x\, \Psi^+\Psi$$

$$= -\sum_{r,s}\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_q}}\left(a_s^+(p)\,u_s^+(p) + b_s(-p)\,v_s^+(-p)\right)\left(a_r(q)\,u_r(q) + b_r^+(-q)\,v_r(-q)\right)\int d^3x\, e^{i(p-q)x}$$

$$= -\sum_{r,s}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\Big(a_s^+(p)u_s^+(p)\,a_r(p)u_r(p) + a_s^+(p)u_s^+(p)\,b_r^+(-p)\,v_r(-p)$$
$$+ b_s(-p)v_s^+(-p)\,a_r(p)\,u_r(p) + b_s(p)\,v_s^+(p)\,b_r^+(p)\,v_r(p)\Big)$$

$$= -\sum_{r,s}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\left(a_s^+(p)\,a_r(p)\,2E_p\,\delta_{rs} + b_s(p)\,b_r^+(p)\,2E_p\,\delta_{rs}\right)$$

$$= -\sum_s\int\frac{d^3p}{(2\pi)^3}\left(a_s^+(p)\,a_s(p) - b_s^+(p)\,b_s(p)\right) + 2\int d^3p\,\delta^3(0)$$

where we exploited the spinor identities

$$u_s^+(p)\,u_r(p) = 2E_p\,\delta_{rs} = v_s^+(p)\,v_r(p) \quad\text{and}\quad u_s^+(p)\,v_r(q) = 0 = v_s^+(p)\,u_r(q)$$

and the anticommutation relation of $b_s(p)$ and $b_r^+(q)$.

$$\{b_s(p), b_r^+(q)\} = (2\pi)^3\,\delta_{rs}\,\delta^{(3)}(\vec{p}-\vec{q}).$$

The second term in Q is clearly divergent but independent of both p and x. It is just an added constant lacking physical meaning. Before applying the charge operator to physical states, this term must be dropped.

d) Show that if $m=0$ the Dirac action admits, in addition, the so-called axial symmetry

$$\Psi(x) \longrightarrow e^{i\alpha\gamma^5}\Psi(x), \qquad \bar{\Psi}(x) \rightarrow \bar{\Psi}(x)e^{i\alpha\gamma^5}, \quad \alpha \in \mathbb{R},$$

which you should interpret as an independent rotation of the positive and negative chirality spinors. Here $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\{\gamma^5, \gamma^\mu\} = 0$ as discussed in the lecture. Deduce the associated conserved current.

For $m=0$, the Dirac Lagrangian reads $\mathcal{L} = i\bar{\Psi}\gamma^\mu\partial_\mu\Psi$. To see how it transforms, we calculate

$$\gamma^\mu e^{i\alpha\gamma^5} = \gamma^\mu \sum_{n=0}^{\infty}\frac{1}{n!}(i\alpha\gamma^5)^n \overset{\{\gamma^5,\gamma^\mu\}=0}{=} \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(i\alpha\gamma^5)^n\gamma^\mu = e^{-i\alpha\gamma^5}\gamma^\mu$$

$$\mathcal{L}' = \bar{\Psi}e^{i\alpha\gamma^5}\gamma^\mu\partial_\mu e^{i\alpha\gamma^5}\Psi = \bar{\Psi}e^{i\alpha\gamma^5}e^{-i\alpha\gamma^5}\gamma^\mu\partial_\mu\Psi = \bar{\Psi}\gamma^\mu\partial_\mu\Psi = \mathcal{L}$$

$\mathcal{L}$ stays invariant and therefore also the action does not change. To find the associated conserved current, we note that

$$\Psi \rightarrow e^{i\alpha\gamma^5}\Psi = \Psi + i\alpha\gamma^5\Psi + \mathcal{O}(\alpha^2) \Rightarrow \delta\Psi = i\gamma^5\Psi, \quad \delta\mathcal{L} = 0 \Rightarrow F^\mu \equiv 0$$

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)}\delta\Psi + \underbrace{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi^+)}}_{0}\delta\Psi^+ - F^\mu = i\bar{\Psi}\gamma^\mu i\gamma^5\Psi = \bar{\Psi}\gamma^5\gamma^\mu\Psi$$

<u>Problem 10.2</u> (Yukawa interaction and Feynman rules)

Consider the following Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi - g\phi\bar{\Psi}\Psi$$

of a massive Dirac fermion Ψ and a massive real scalar ϕ coupled by a Yukawa interaction with a real coupling constant g.

a) Deduce the mass dimension of the coupling g.

In natural units, we set $\hbar = 1$. Therefore, the action S has mass dimension zero. From this we can deduce dimensions of other quantities.

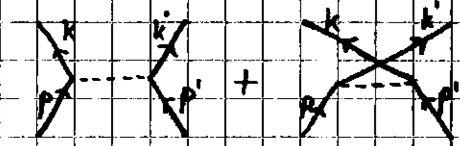
$$0 = [S] = [d^4x\,\mathcal{L}] = -4 + [\mathcal{L}] \implies [\mathcal{L}] = 4$$

$$4 = [\mathcal{L}] = [m^2 \phi^2] = 2 + [\phi^2] \implies [\phi] = 1$$

$$4 = [\mathcal{L}] = [\overline{\Psi} m \Psi] = 1 + [\Psi^2] \implies [\Psi] = \frac{3}{2}$$

$$4 = [\mathcal{L}] = [g \phi \overline{\Psi} \Psi] = [g] + 1 + 2\,\frac{3}{2} \implies [g] = 0$$

b) In the previous exercise, we discussed a quick way to compute the the S-matrix to leading order in the coupling constant directly in the quantum mechanical interaction picture. Use this procedure to compute, from scratch, the complete leading order scattering amplitude for scattering of two fermions with ingoing momenta p and p' in Yukawa theory. The result is:

$i\mathcal{M} =$ 

$$= (-ig)^2 \left(\bar{u}(k) u(p) \frac{1}{(k-p)^2 - m_\phi^2} \bar{u}(k') u(p') - \bar{u}(k) u(p') \frac{1}{(k-p')^2 - m_\phi^2} \bar{u}(k') u(p) \right) \quad (6)$$

This exercise is far too lengthy for the exam and therefore does not need to be considered further at this point.

Problem 10.3 (Parity)

Parity is the discrete map $P : x \longrightarrow x' = Px$, $P(x^0, \vec{x}) = (x^0, -\vec{x})$. To find the transformation of the Dirac spinor field $\Psi(x)$ under parity transformations we seek for a representation of the group of parity transformations

$$\Psi(x) \longrightarrow \Psi'(x) := P\, \Psi(x) P^{-1} := S(P)\, \Psi(P^{-1}x), \quad (8)$$

where $S(P)$ acts on the Dirac spinor.

a) Deduce the necessary condition for $S(P)$ such that the Dirac equation is invariant under parity transformation and verify on the basis of the Clifford algebra that $S(P) = \gamma^0$ satisfies these.

We take this as a definition of the parity transformed spinor field, i.e. $\Psi(x)$ transforms as

$$\Psi(x) \longrightarrow \eta \gamma^0 \Psi(P^{-1}x'), \quad \eta^2 = 1, \text{ where we set } \eta = 1. \tag{10}$$

The Dirac equation $(i\gamma^\mu \partial_\mu - m)\Psi(x) = 0$ is obtained by varying the action of the free classical Dirac field

$$S = \int d^4x\, \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi$$

with respect to $\Psi^\dagger$. Demanding that the Dirac equation stay invariant under parity transformation results in

$$(i\gamma^\mu \partial_\mu - m)\Psi(x) = 0 \xrightarrow{P} (i\gamma^\mu \partial^\mu - m) S(P)\, \Psi(P^{-1}x') = 0$$

Here, it shouldn't make any difference if we first transform the spinor and then act on it with the transformed operators in the Dirac equation or first let the old operators act on the old spinor and then transform the so obtained spinor, i.e. we can further claim

$$(i\gamma^\mu \partial^\mu - m) S(P)\, \Psi(P^{-1}x') \stackrel{!}{=} S(P)(i\gamma^\mu \partial_\mu - m)\Psi(P^{-1}x')$$

We have essentially commuted $S(P)$ with γ^μ and found that this gives us a minus sign in the three spatial components. That is akin to saying $S(P)$ fulfills the following relations

$$[S(P), \gamma^0] = 0, \qquad \{S(P), \gamma^i\} = 0$$

Incidentally, thanks to the defining relation of the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}_4$, we find that γ^0 satisfies the above. In general, $S(P)$ is therefore given by

$$S(P) = \eta\gamma^0, \text{ where we concretize by setting } \eta = 1.$$

b) Show that parity flips chirality, i.e.

$$\mathbb{P}_{\pm}\left(S(P)\,\Psi_{\pm}(P^{-1}x)\right) = 0, \qquad \mathbb{P}_{\mp}\left(S(P)\,\Psi_{\pm}(P^{-1}x)\right) = S(P)\,\Psi_{\pm}(P^{-1}x), \tag{11}$$

where $\mathbb{P}_{\pm} = \frac{1}{2}(\mathbb{1}_4 \pm \gamma^5)$ is the projection operator onto chiral and anti-chiral spinors $\Psi_{\pm} = \mathbb{P}_{\pm}\Psi$. You may use the above representation in eq. (10).

Since $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}_4$, γ^0 anticommutes with γ^i, where $i \in \{1,2,3\}$

$$\{\gamma^0, \gamma^i\} = 2\eta^{0i}\,\mathbb{1}_4 = 0$$

From this, we can deduce that also γ^0 and γ^5 anticommute

$$\{\gamma^0, \gamma^5\} = \gamma^0 i\gamma^0\gamma^1\gamma^2\gamma^3 + i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0 = \gamma^0 i\gamma^0\gamma^1\gamma^2\gamma^3 + (-1)^3\gamma^0 i\gamma^0\gamma^1\gamma^2\gamma^3 = 0$$

Since we are allowed to use eq. (10), we can write

$$\mathbb{P}_{\pm}(S(P)\,\Psi_{\pm}(P^{-1}x)) = \frac{1}{2}(\mathbb{1}_4 \pm \gamma^5)\,\gamma^0\,\Psi_{\pm}(P^{-1}x) = \gamma^0\,\frac{1}{2}(\mathbb{1}_4 \mp \gamma^5)\,\Psi_{\pm}(P^{-1}x)$$

$$= \gamma^0\,\mathbb{P}_{\mp}\,\Psi_{\pm}(P^{-1}x) = \gamma^0\,\mathbb{P}_{\mp}\,\mathbb{P}_{\pm}\,\Psi(P^{-1}x) = 0$$

$$\mathbb{P}_{\mp}(S(P)\,\Psi_{\pm}(P^{-1}x)) = \frac{1}{2}(\mathbb{1}_4 \mp \gamma^5)\,\gamma^0\,\Psi_{\pm}(P^{-1}x) = \gamma^0\,\frac{1}{2}(\mathbb{1}_4 \pm \gamma^5)\,\Psi_{\pm}(P^{-1}x)$$

$$= \gamma^0\,\mathbb{P}_{\pm}\,\Psi_{\pm}(P^{-1}x) = \gamma^0\,\mathbb{P}_{\pm}\,\mathbb{P}_{\pm}\,\Psi(P^{-1}x) = \gamma^0\,\mathbb{P}_{\pm}\,\Psi(P^{-1}x) = S(P)\,\Psi_{\pm}(P^{-1}x)$$

c) Determine the transformation behaviour of the spinor bilinears $\bar\Psi(x)\,\Psi(x)$, $\bar\Psi(x)\,\gamma^5\Psi(x)$, $\bar\Psi(x)\,\gamma^\mu\,\Psi(x)$, and $\bar\Psi(x)\,\gamma^\mu\gamma^5\,\Psi(x)$ under parity transformations and also under continuous Lorentz transformations.

Note: The object $\bar\Psi(x)\,\gamma^5\,\Psi(x)$ is called a pseudo-scalar and the object $\bar\Psi(x)\,\gamma^\mu\gamma^5\,\Psi(x)$ is called a pseudo-vector or axial vector due to their properties under parity transformations.

$$\bar\Psi(x)\,\Psi(x) \xrightarrow{P} \overline{S(P)\,\Psi(P^{-1}x)}\;S(P)\,\Psi(P^{-1}x) = (S(P)\,\Psi(P^{-1}x))^\dagger\underbrace{\gamma^0\gamma^0}_{\mathbb{1}_4}\Psi(P^{-1}x)$$

$$= \Psi^\dagger(P^{-1}x)\,S^\dagger(P)\,\Psi(P^{-1}x) = \Psi^\dagger(P^{-1}x)\underbrace{(\gamma^0)^\dagger}_{\gamma^0}\Psi(P^{-1}x) = \bar\Psi(P^{-1}x)\,\Psi(P^{-1}x)$$

$$\overline{\Psi}(x)\Psi(x) \xrightarrow{\Lambda} \overline{S(\Lambda)\Psi(\Lambda^{-1}x')}\, S(\Lambda)\Psi(\Lambda^{-1}x') = \Psi^\dagger(\Lambda^{-1}x')\, S^\dagger(\Lambda)\, \gamma^0 S(\Lambda)\Psi(\Lambda^{-1}x')$$

$$= \Psi^\dagger(\Lambda^{-1}x')\, \underbrace{\gamma^0\gamma^0}_{1_4}\, \underbrace{S^\dagger(\Lambda)\gamma^0}_{S^{-1}(\Lambda)} S(\Lambda)\Psi(\Lambda^{-1}x') = \overline{\Psi}(\Lambda^{-1}x')\Psi(\Lambda^{-1}x')$$

$$\overline{\Psi}(x)\gamma^5\Psi(x) \xrightarrow{P} \Psi^\dagger(P^{-1}x')\, S^\dagger(P)\gamma^0\gamma^5 S(P)\Psi(P^{-1}x') = \Psi^\dagger(P^{-1}x')\underbrace{\gamma^0\gamma^0}_{1_4}\gamma^5\gamma^0\Psi(P^{-1}x')$$

$$= -\Psi^\dagger(P^{-1}x')\gamma^0\gamma^5\Psi(P^{-1}x') = -\overline{\Psi}(P^{-1}x')\gamma^5\Psi(P^{-1}x')$$

$$\overline{\Psi}(x)\gamma^5\Psi(x) \xrightarrow{\Lambda} \Psi^\dagger(\Lambda^{-1}x)\, S^\dagger(\Lambda)\gamma^0\gamma^5 S(\Lambda)\Psi(\Lambda^{-1}x) = \Psi^\dagger(P^{-1}x')\gamma^0 S^{-1}(\Lambda)\gamma^5 S(\Lambda)\Psi(\Lambda^{-1}x')$$

$$= \overline{\Psi}(P^{-1}x')\gamma^5 S^{-1}(\Lambda) S(\Lambda)\Psi(\Lambda^{-1}x) = \overline{\Psi}(P^{-1}x')\gamma^5\Psi(\Lambda^{-1}x')$$

where we used that $[S^{-1}(\Lambda),\gamma^5]=0$ since $S^{-1}(\Lambda) = e^{\frac{i}{2}\omega_{\rho\nu}S^{\rho\nu}}$ and $S^{\mu\nu} = \frac{i}{2}[\gamma^\mu\gamma^\nu - \eta^{\mu\nu}]$

$$\overline{\Psi}(x)\gamma^\mu\Psi(x) \xrightarrow{P} \Psi^\dagger(P^{-1}x')\, S^\dagger(P)\gamma^0\gamma^\mu S(P)\Psi(P^{-1}x') = \Psi^\dagger(P^{-1}x')\underbrace{\gamma^0\gamma^0}_{1_4}\gamma_\mu\Psi(P^{-1}x')$$

$$\overline{\Psi}(x)\gamma^\mu\Psi(x) \xrightarrow{\Lambda} \Psi^\dagger(\Lambda^{-1}x')\, S^\dagger(\Lambda)\gamma^0\gamma^\mu S(\Lambda)\Psi(\Lambda^{-1}x') = \overline{\Psi}(\Lambda^{-1}x')\underbrace{S^{-1}(\Lambda)\gamma^\mu S(\Lambda)}_{\Lambda^\mu{}_\nu\gamma^\nu}\Psi(\Lambda^{-1}x')$$

$$\overline{\Psi}(x)\gamma^\mu\gamma^5\Psi(x) \xrightarrow{P} \Psi^\dagger(P^{-1}x')\, S^\dagger(P)\gamma^0\gamma^\mu\gamma^5 S(P)\Psi(P^{-1}x') = \Psi^\dagger(P^{-1}x')\gamma^\mu\gamma^5\gamma^0\Psi(P^{-1}x')$$

$$= -\Psi^\dagger(P^{-1}x')\gamma^0\gamma_\mu\gamma^5\Psi(P^{-1}x') = -\overline{\Psi}(P^{-1}x')\gamma_\mu\gamma^5\Psi(P^{-1}x')$$

$$\overline{\Psi}(x)\gamma^\mu\gamma^5\Psi(x) \xrightarrow{\Lambda} \Psi^\dagger(\Lambda^{-1}x')\, S^\dagger(\Lambda)\gamma^0\gamma^\mu\gamma^5 S(\Lambda)\Psi(\Lambda^{-1}x') = \overline{\Psi}(\Lambda^{-1}x')\, S^{-1}(\Lambda)\gamma^\mu\gamma^5 S(\Lambda)\Psi(\Lambda^{-1}x')$$

$$= \overline{\Psi}(\Lambda^{-1}x')\,\Lambda^\mu{}_\nu\gamma^\nu\gamma^5\Psi(\Lambda^{-1}x')$$

Problem 11.1 (The Yukawa potential)

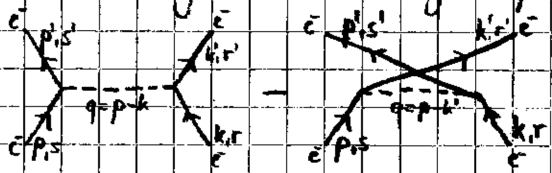
Here, we will analyse and 'derive' the classical potential for Yukawa theory, where the interactions between fermions are transmitted by a scalar field ϕ. As discussed on assignment 10 in problem 10.2, the Lagrangian of this theory is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \overline{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi - g\,\phi\,\overline{\Psi}\Psi. \qquad (1)$$

The Feynman rules are given in the script. Now analyse the following two processes

a) Electron-electron scattering: $e^- + e^- \rightarrow e^- + e^-$.

At tree-level, there are just two diagrams contributing to electron-electron scattering. The scattering amplitude is given by

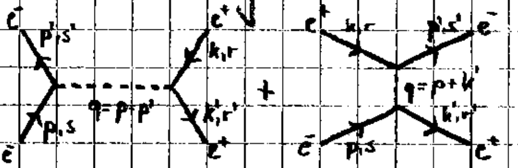
$i\mathcal{M} =$ 

$$= \bar{u}_{s'}(p') u_s(p) (-ig) \frac{i}{(p-p')^2 - m_\phi^2} (-ig)\, \bar{u}_{r'}(k') u_r(k) - \bar{u}_{r'}(k) u_s(p) (-ig) \frac{i}{(p-k')^2 - m_\phi^2} (-ig)\, \bar{u}_{s'}(p') u_r(k)$$

$$= -ig^2 \left(\bar{u}_{s'}(p') u_s(p) \frac{1}{(p-p')^2 - m_\phi^2} \bar{u}_{r'}(k') u_r(k) - \bar{u}_{r'}(k) u_s(p) \frac{1}{(p-k)^2 - m_\phi^2} \bar{u}_{s'}(p') u_r(k) \right)$$

b) Electron-positron scattering: $e^- + e^+ \rightarrow e^- + e^+$

This so-called Bhabha scattering also has two diagrams contributing to the scattering amplitude at tree-level. While the second diagram from part a) is no longer an option because the particles are no longer indistinguishable. Instead, we can now have electron-positron annihilation, i.e. an s-channel diagram.

$i\mathcal{M} =$ 

$$= \bar{u}_{s'}(p') u_s(p) (-ig) \frac{i}{(p-p')^2 - m_\phi^2} (-ig)\, \bar{v}_{r'}(k') v_r(k) + \bar{v}_{r'}(k') u_s(p) (-ig) \frac{i}{(p+k')^2 - m_\phi^2} (-ig)\, \bar{u}_{s'}(p') v_r(k)$$

$$= -ig^2 \left(\bar{u}_{s'}(p') u_s(p) \frac{1}{(p-p')^2 - m_\phi^2} \bar{v}_{r'}(k') v_r(k) + \bar{v}_{r'}(k') u_s(p) \frac{1}{(p+k')^2 - m_\phi^2} \bar{u}_{s'}(p') v_r(k) \right)$$

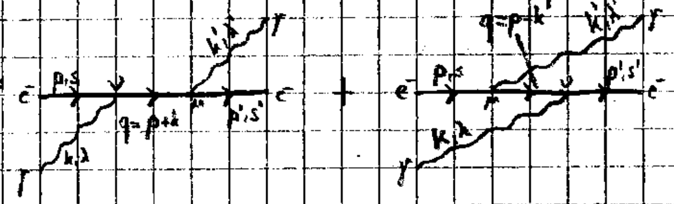
Problem 11.2 (The Klein-Nishima-formula for Compton scattering)

We now look at the process of compton scattering at tree-level in QED. The Feynman rules for QED are listed in the script in section 4.6.

a) Use the Feynman rules write down the tree-level Compton scattering amplitude and bring it into the form

$$i\mathcal{M} = -i e^2 \epsilon^*_\mu(k',\lambda') \epsilon_\nu(k,\lambda)\, \bar{u}_{s'}(p') \left(\frac{\gamma^\mu \not{k} \gamma^\nu + 2\gamma^\mu p^\nu}{2pk} + \frac{\gamma^\nu \not{k}' \gamma^\mu - 2\gamma^\nu p^\mu}{2pk'} \right) u_s(p).$$

At tree-level, Compton scattering can take place in the form of two distinct processes. In $e^- + \gamma \longrightarrow e^- + \gamma$, the electron can either first absorb, then emit a photon, making it an s-channel diagram or first emit, then absorb a photon, which results in t-channel scattering.

$i\mathcal{M} =$ 

$$= \bar{u}_{s'}(p')\, \varepsilon^*_\mu(k',\lambda')\,(-ie\gamma^\mu)\, \frac{i(\gamma(p+k)+m_0)}{(p+k)^2-m_0^2}\,(-ie\gamma^\nu)\, u_s(p)\, \varepsilon_\nu(k,\lambda)$$

$$+ \bar{u}_{s'}(p')\, \varepsilon_\nu(k,\lambda)\,(-ie\gamma^\nu)\, \frac{i(\gamma(p-k')+m_0)}{(p-k')^2-m_0^2}\,(-ie\gamma^\mu)\, u_s(p)\, \varepsilon^*_\mu(k,\lambda')$$

$$= -ie^2\, \varepsilon^*_\mu(k',\lambda')\, \varepsilon_\nu(k,\lambda)\, \bar{u}_{s'}(p') \left(\gamma^\mu \frac{\not{p}+\not{k}+m_0}{2pk} \gamma^\nu + \gamma^\nu \frac{\not{p}-\not{k}'+m_0}{-2pk'} \gamma^\mu \right) u_s(p)$$

Using the Dirac equation and the anticommutation relation of the Clifford algebra, $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu + 2\eta^{\mu\nu}\mathbb{1}_4$, we can simplify the numerators.

$$(\not{p} + m_0)\gamma^\nu u_s(p) = (\gamma^\mu p_\mu + m_0)\gamma^\nu u_s(p) = [2\eta^{\mu\nu}\mathbb{1}_4 p_\mu + \gamma^\nu \underbrace{(-\gamma^\mu p_\mu + m_0)] u_s(p)}_{=0,\ \text{e.o.m.}} = 2p^\nu$$

Using this equality, we can write the scattering amplitude $i\mathcal{M}$ as

$$i\mathcal{M} = -ie^2\, \varepsilon^*_\mu(k',\lambda')\, \varepsilon(k,\lambda)\, \bar{u}_{s'}(p') \left(\gamma^\mu \frac{\not{k}\gamma^\nu + 2p^\nu}{2pk} + \gamma^\nu \frac{\not{k}'\gamma^\mu - 2p^\mu}{2pk'} \right) u_s(p)$$

Parts b) through g) are too technical for the exam and will thus not be solved here.