Revision of Exercises somewhat relevant for the Exam

Problem 1.1 (Lorentz transformation)

A Lorentz transformation acts on spacetime 4-vectors x^μ with $\mu \in \{0, i\}$ and $i \in \{1,2,3\}$ as

$$x^\mu \longrightarrow x^{\mu'} = \Lambda^\mu{}_\nu x^\nu$$

such that $x^2 = x^\mu x^\nu x_{\mu\nu}$ is invariant with $\mathrm{diag}(\eta_{\mu\nu}) = (1, -1, -1, -1)$.

a) Given two spacetime points x^μ and y^μ, show that if $(x-y)^2 > 0$, one can always find a frame in which the spatial distance vanishes, $x^i = y^i$. Show that if $(x-y)^2 < 0$, one can always find a frame where $x^0 = y^0$.

The defining relation of the Lorentz algebra $so(1,3)$ is

$$\Lambda^\alpha{}_\beta \Lambda^\gamma{}_\delta \eta_{\alpha\gamma} = \eta_{\beta\delta}, \quad \text{where } \eta_{\alpha\gamma} = \eta_{\beta\delta},$$

i.e. $so(1,3)$ is the set of all transformations that leave the metric invariant. For simplicity, we now consider a coordinate system in which it holds that

$$\Delta^\mu = x^\mu - y^\mu = (\Delta^0, \vec{\Delta}) \equiv (\Delta^0, \Delta^1, 0, 0)$$

We can then reduce the problem to two dimensions in which we transform by

$$\eta_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \Lambda^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Lambda = \Lambda^T \eta_2 \Lambda, \qquad \text{with } \Lambda = \begin{pmatrix} \cosh(\alpha) & -\sinh(\alpha) \\ -\sinh(\alpha) & \cosh(\alpha) \end{pmatrix},$$

which gives in the case of Δ

$$\Delta_2' = \begin{pmatrix} \Delta^{0'} \\ \Delta^{1'} \end{pmatrix} = \begin{pmatrix} \cosh(\alpha) & -\sinh(\alpha) \\ -\sinh(\alpha) & \cosh(\alpha) \end{pmatrix} \begin{pmatrix} \Delta^0 \\ \Delta^1 \end{pmatrix} = \begin{pmatrix} \Delta^0 \cosh(\alpha) - \Delta^1 \sinh(\alpha) \\ -\Delta^0 \sinh(\alpha) + \Delta^1 \cosh(\alpha) \end{pmatrix} = \Lambda \Delta$$

We now have a system of equations in which we can set $\Delta^{0'} = \Delta^{1'} = 0$ to see what conditions arise for Δ^0 and Δ^1 when we demand x_2 and y_2 to occur at the same time and place respectively.

$$\Delta^{0'} = 0: \quad \Delta^0 \cosh(\alpha) = \Delta^1 \sinh(\alpha) \quad \Longrightarrow \quad \Delta^0 = \tanh(\alpha) \Delta^1 \tag{1}$$

$$\Delta^{1'} = 0: \quad \Delta^0 \sinh(\alpha) = \Delta^1 \cosh(\alpha) \quad \Longrightarrow \quad \Delta^1 = \tanh(\alpha) \Delta^0 \tag{2}$$

Since $\tanh(\alpha) \in (-1,1)\ \forall \alpha \in \mathbb{R}$ and therefore in particular $\tanh(\alpha) < 1\ \forall \alpha \in \mathbb{R}$, we have found that if $(x-y)^2 > 0$ or rather $(x_1 - y_1)^2 = \Delta^2 = (\Delta^0)^2 - (\Delta^1)^2 > 0$, i.e. $\Delta^0 > \Delta^1$, we can indeed find such an α that fulfills (1)

$$\Delta^0 = \tanh(\alpha)\, \Delta^1$$

which implies $\Delta^{0\prime} = 0$. In other words, if $(x-y) > 0$ there exists such an α that the reference frame reached by transforming with $\Lambda(\alpha)$ observes x and y to occur at the same time. However, there can be no $\alpha \in \mathbb{R}$ such that $\Delta^1 = \tanh(\alpha)\Delta^0$ in this setup. On the other hand, if we look at $(x-y)^2 < 0$ which amounts to $\Delta^0 < \Delta^1$, there can be no α to fulfill (1) but instead (2) is possible suggesting that for this setup, there is always a frame in which x and y happen at the same place.

b) Given a scalar field theory with Lagrangian $\mathcal{L}(\phi(x), \partial_\mu \phi(x))$, show that the object $\frac{\partial \mathcal{L}(x)}{\partial(\partial_\mu \phi(x))}$ transforms like a 4-vector under Lorentz transformations.
We show this by simply applying a Lorentz transformation to this expression, where we keep in mind that $\mathcal{L}$ is necessarily a Lorentz scalar. We define $\partial_{\mu'} = \frac{\partial}{\partial x'^{\mu}}$

$$\frac{\partial \mathcal{L}(x)}{\partial(\partial_\mu \phi(x))} \longrightarrow \frac{\partial \mathcal{L}(x')}{\partial(\partial_{\mu'} \phi(x'))} = \frac{\partial \mathcal{L}(x')}{\partial(\partial_\mu \phi(x'))} \frac{\partial(\partial_\mu \phi(x'))}{\partial(\partial_{\mu'} \phi(x'))} = \frac{\partial \mathcal{L}(x')}{\partial(\partial_\mu \phi(x'))} \frac{\partial}{\partial(\partial_{\mu'} \phi(x'))} \underbrace{\frac{\partial x'^{\nu}}{\partial x^{\mu}}}_{\text{constant}} \frac{\partial \phi(x')}{\partial x'^{\nu}}$$

$$= \underbrace{\frac{\partial x'^{\nu}}{\partial x^{\mu}}}_{\Lambda^{\nu}{}_{\mu}} \frac{\partial \mathcal{L}(x')}{\partial(\partial_\mu \phi(x'))} \underbrace{\frac{\partial(\partial_{\nu'} \phi(x'))}{\partial(\partial_{\mu'} \phi(x'))}}_{\delta_{\mu'\nu'}} = \Lambda^{\mu'}{}_{\mu} \frac{\partial \mathcal{L}(x')}{\partial(\partial_\mu \phi(x'))}$$

Problem 10.1 (LSZ versus Interaction picture perturbation theory)
We would like to compare the exact LSZ result for the scattering matrix with a more naive approach which is inspired by Quantum Mechanical perturbation theory in the Interaction picture.
We model the in and out-going states as free particle states in the Interaction picture

$$|p_I\rangle = \sqrt{2E_p}\ a_I^\dagger(p)|0\rangle,$$

where $a_I^\dagger(p)$ are the modes of the free Interaction picture fields and $|0\rangle$ is the free vacuum in the Interaction picture. The time-evolution of such Interaction picture states is governed by the time-evolution operator $U(t,t_0)$ such that

$$|p_I(t)\rangle = U(t,t_0)\,|p_I(t_0)\rangle, \qquad \text{where} \quad U(t,t_0) = T\exp\left(-i\int_{t_0}^{t} H_I(t')\,dt'\right).$$

The scattering amplitude is then given by

$$\langle p_I^1, \dots, p_I^n|\,T\exp\left(-i\int_{-\infty}^{\infty} H_I(t')\,dt'\right)|q_I^1, \dots, q_I^r\rangle.$$

a) Critise this prescription from your understanding of the exact LSZ result and the underlying theory of asymptotic in- and outstates.

Asymptotic in- and outstates derive from the full interacting theory. Consequently, while not interacting with each other for $t \to \pm\infty$, they perform selfinteractions, thereby accounting for central features of the interacting theory such as the wavefunction renormalization $Z = 1 + \mathcal{O}(\lambda)$ and the fully renormalized mass $m \neq m_0$. In the Interaction picture perturbation theory approach, in- and outgoing states are modelled as free particle states which discards of the above mentioned corrections. Moreover, these states are created from the free vacuum $|0\rangle$ as opposed to the vacuum of the full interacting theory $|\Omega\rangle$ used in the LSZ formalism.

To leading order in the interacting theory's coupling, both formalisms agree because

- we consider only amputated, fully connected diagrams, where the amputated means that we have no factors of $\frac{1}{p^2 - m^2}$ in which the renormalized mass $m \neq m_0$ could show up.
- both $|\Omega\rangle$ and Z are of the form $|\Omega\rangle = (1 + \mathcal{O}(\lambda))|0\rangle$ and $Z = 1 + \mathcal{O}(\lambda)$, so that to leading order only the 1 contributes.

b) In the Interaction picture approach, we can perform leading-order scattering amplitude computations very explicitly and from scratch. Compute the amplitude for 2-2 scattering in ϕ^4-theory in this fashion by evaluating

$$\langle p_I^1, p_I^2 | \, T \int d^4x \frac{-i\lambda}{4!} \phi_I^4(x) | q_I^1, q_I^2 \rangle .$$

Hint: Expand $\phi_I(x) = \phi_I^+(x) + \phi_I^-(x)$. Argue that only terms with precisely two copies of $\phi_I^+(x)$ and $\phi_I^-(x)$ contribute and evaluate the matrix elements by commuting creation and annihilation operators appropriately.

We first note that when applying Wick's theorem, we need not consider internal contractions since we are looking specifically at nontrivial scattering of four particles by one vertex that can have only four connections.

$$T\phi_I^4(x) = :\phi_I^4(x): + \text{'terms with internal contractions (will be dropped henceforth)'}$$

$$= :\phi_I^+\phi_I^+\phi_I^+\phi_I^+ + \phi_I^+\phi_I^+\phi_I^+\phi_I^- + \phi_I^+\phi_I^+\phi_I^-\phi_I^+ + \phi_I^+\phi_I^+\phi_I^-\phi_I^- + \phi_I^+\phi_I^-\phi_I^+\phi_I^+$$

$$+ \phi_I^+\phi_I^-\phi_I^+\phi_I^- + \phi_I^+\phi_I^-\phi_I^-\phi_I^+ + \phi_I^+\phi_I^-\phi_I^-\phi_I^- + \text{'same again with } \phi_I^- \text{ in front'}:$$

As hinted, we consider only those of these 16 terms which contain each two ϕ_I^+ and ϕ_I^-. We can understand why this is legitimate from looking at how ϕ_I^+ and $a^+(p)$ commute.

$$[\phi_I^+(x), a^+(p)] = \int \frac{d^3q}{(2\pi)^3} \frac{e^{-iqx}}{\sqrt{2E_q}} [a(q), a^+(p)] = \int \frac{d^3q}{(2\pi)^3} \frac{e^{-iqx}}{\sqrt{2E_q}} (2\pi)^3 \delta^{(3)}(\vec{q}-\vec{p}) = \frac{e^{-ipx}}{\sqrt{2E_p}} =: \frac{\alpha_p}{\sqrt{2E_p}}$$

$$\phi_I^+(x)\phi_I^+(x)|p_I^1, p_I^2\rangle = \phi_I^+(x)\phi_I^+(x)\sqrt{2E_{p_1}}\sqrt{2E_{p_2}}\, a_I^+(p_1)\, a_I^+(p_2)|0\rangle$$

$$= \sqrt{2E_{p_1}}\sqrt{2E_{p_2}}\, \phi_I^+(x)\, a_I^+(p_1)\, \phi_I^+(x)\, a_I^+(p_2)|0\rangle + \alpha_{p_1}\sqrt{2E_{p_2}}\, \phi_I^+(x)\, a_I^+(p_2)|0\rangle$$

$$= \sqrt{2E_{p_1}}\sqrt{2E_{p_2}}\, \phi_I^+(x)\, a_I^+(p_1)\, a_I^+(p_2) \underbrace{\phi_I^+(x)|0\rangle}_{0} + \alpha_{p_2}\sqrt{2E_{p_1}}\, \phi_I^+(x)\, a_I^+(p_1)|0\rangle + \alpha_{p_1}\sqrt{2E_{p_2}}\, a_I^+(p_2) \underbrace{\phi_I^+(x)|0\rangle}_{0}$$

$$+ \alpha_{p_1}\alpha_{p_2}|0\rangle = \alpha_{p_2}\sqrt{2E_{p_1}}\, a_I^+(p_1) \underbrace{\phi_I^+(x)|0\rangle}_{0} + \alpha_{p_1}\alpha_{p_2}|0\rangle + \alpha_{p_1}\alpha_{p_2}|0\rangle = 2\alpha_{p_1}\alpha_{p_2}|0\rangle$$

Now since $\alpha_i = e^{-ip_ix} \in \mathbb{C}$ ($\notin$ state space), we see that as soon as we act on the state $|p_I^1, p_I^2\rangle$ with another set of positive frequency modes, it vanishes. This also holds true for the state $\langle p_I^1, p_I^2|$ acted upon by $\phi_I^-(x)\phi_I^-(x)$ because

$$\langle p_I^1, p_I^2 | \phi_I^-(x)\phi_I^-(x) = \left(\phi_I^+(x)\phi_I^+(x)|p_I^1, p_I^2\rangle\right)^+ = \left(2\alpha_{p_1}\alpha_{p_2}|0\rangle\right)^+ = \langle 0| 2\alpha_{p_1}^*\alpha_{p_2}^*$$

As soon as we act on $\langle p_I^1, p_I^2|$ with a third $\phi_I(x)$, it too vanishes.

We therefore have for the time-ordered product of $\phi^4(x)$

$$T\phi_I^4(x) = :\phi_I^+\phi_I^-\phi_I^+\phi_I^- + \phi_I^+\phi_I^-\phi_I^-\phi_I^+ + \phi_I^-\phi_I^+\phi_I^+\phi_I^- + \phi_I^-\phi_I^+\phi_I^-\phi_I^+ + \phi_I^-\phi_I^-\phi_I^+\phi_I^+$$

$$+\phi_I^+\phi_I^+\phi_I^-\phi_I^-: + \text{'terms not contributing to scattering amplitude'}$$

$$= 6\,\phi_I^-(x)\phi_I^-(x)\phi_I^+(x)\phi_I^+(x) + \text{'terms not contributing to scattering amplitude'}$$

Inserting this finding into the expression we would like to evaluate gives

$$\langle p_I^1, p_I^2| T\int d^4x \frac{-i\lambda}{4!}\phi_I^4(x)|q_I^1, q_I^2\rangle = -\frac{i\lambda}{4!}\int d^4x \langle p_I^1, p_I^2| 6\,\phi_I^-(x)\phi_I^-(x)\phi_I^+(x)\phi_I^+(x)|q_I^1, q_I^2\rangle$$

$$= -\frac{i\lambda}{4}\int d^4x \langle 0| 2\alpha_{p_1}^*\alpha_{p_2}^* 2\alpha_{q_1}\alpha_{q_2}|0\rangle = -i\lambda\int d^4x\, e^{ip_1x}e^{ip_2x}e^{-iq_1x}e^{-iq_2x}\underbrace{\langle 0|0\rangle}_{1}$$

$$= -i\lambda\,(2\pi)^4\delta^{(4)}(p_1+p_2-q_1-q_2),$$ where $\mathcal{M} = -\lambda$ is the actual scattering amplitude

Problem 10.2 (Yukawa interactions and Feynman rules)

Consider the following Lagrangian

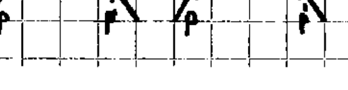
$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m_\phi^2\phi^2 + \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi - g\phi\bar{\Psi}\Psi$$

of a massive Dirac fermion Ψ and a massive real scalar field ϕ coupled by what is called a Yukawa interaction with a real coupling constant g.

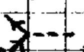
a) Deduce the mass dimension of the coupling g.

Evidently $[S] = 0 \rightarrow [\mathcal{L}] = 4 \rightarrow [\phi] = 1, [\Psi] = \frac{3}{2} \rightarrow [g] = 0$.

b) In the previous exercise, we discussed a quick way to compute the S-matrix to leading order in the coupling constant directly in the quantum mechanical Interaction picture. Use this procedure to compute, from scratch, the complete leading order scattering amplitude for the scattering of two fermions with ingoing momenta p and p' in Yukawa theory. The result is

$i\mathcal{M} =$ 

$= (-ig)^2\left(\bar{u}(k)u(p)\frac{1}{(k-p)^2-m_\phi^2}\bar{u}(k')u(p') - \bar{u}(k)u(p')\frac{1}{(k-p')^2-m_\phi^2}\bar{u}(k')u(p)\right).$

By taking p, p' and k, k' to be distinct, we effectively exclude zero-vertex diagrams, i.e. $\langle k, k' | p, p' \rangle = 0$. From looking at the Yukawa interaction $\mathcal{L}_{int} = -g\phi\bar{\Psi}\psi$ however, it is immediately obvious that one-vertex diagrams, i.e. , cannot produce this type of scattering either. Leading order in this case therefore means second order in the coupling constant.

$$S_{fin} = \langle k, k' | U(\infty, -\infty) | p, p' \rangle = \langle k, k' | T \exp\left(-i\int_{-\infty}^{\infty} H_I(t)\, dt\right) | p, p' \rangle = \underbrace{\langle k, k' | T\, \mathbb{1} | p, p' \rangle}_{0}$$

$$+ \underbrace{\langle k, k' | T \left(-i\int_{-\infty}^{\infty} H_I(t)\, dt\right) | p, p' \rangle}_{0} + \langle k, k' | T \frac{1}{2!}\left(-i\int_{-\infty}^{\infty} H_I(t)\, dt\right)^2 | p, p' \rangle + \sigma(\lambda^3)$$

where the integral over time can be expanded to an integral over spacetime via

$$\int_{-\infty}^{\infty} H_I(t)\, dt = \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \mathcal{H}_I(t, \vec{x})\, d^3x\, dt = \int d^4x\, e^{iH_0(t-t_0)} \underbrace{\mathcal{H}_{int}}_{-\mathcal{L}_{int}} e^{-iH_0(t-t_0)}$$

$$= g\int d^4x\, e^{iH_0(t-t_0)}\phi\, e^{-iH_0(t-t_0)} e^{iH_0(t-t_0)} \bar{\Psi} e^{-iH_0(t-t_0)} e^{iH_0(t-t_0)} \Psi e^{-iH_0(t-t_0)} = g\int d^4x\, \phi_I \bar{\Psi}_I \Psi_I$$

Inserting the integral over Interaction picture into $S = S_{fin} - \sigma(\lambda^3)$ gives

$$S = \frac{1}{2}(-ig)^2 \int d^4x \int d^4y\, \langle k, k' | T\, \phi_I(x) \bar{\Psi}_I(x) \Psi_I(x) \phi_I(y) \bar{\Psi}_I(y) \Psi_I(y) | p, p' \rangle \qquad (1)$$

We now apply Wick's theorem to the above time-ordered product of six fields, where we keep in mind that terms relevant for our type of scattering need to contain exactly four uncontracted fermions and zero uncontracted scalars. For brevity, we write $\phi_I(x)$, $\bar{\Psi}_I(x)$ and $\Psi_I(x)$ as ϕ_x, $\bar{\Psi}_x$, and Ψ_x.

$$T\,\phi_x \bar{\Psi}_x \Psi_x \phi_y \bar{\Psi}_y \Psi_y = :\overbrace{\phi_x \phi_y}\, \bar{\Psi}_x \Psi_x \bar{\Psi}_y \Psi_y: + \text{'terms not contributing to our type of scat.'}$$

In the only relevant term of the Wick expansion, the contracted scalar fields form a propagator: $\overbrace{\phi_x \phi_y} = D_F^{\phi}(x-y)$, which is merely a complex number and can therefore be pulled out of the normal-ordering. When normal-ordering the four remaining fermionic fields, we get a total of 16 terms. Of these, we again need to consider only those that contain precisely two copies of Ψ^+ and Ψ^-.

$$
\begin{aligned}
:\bar\Psi_x \psi_x \bar\Psi_y \psi_y: = :\, & \bar\Psi^+_x \psi^+_x \bar\Psi^+_y \psi^+_y + \bar\Psi^+_x \psi^+_x \bar\Psi^+_y \psi^-_y + \bar\Psi^+_x \psi^+_x \bar\Psi^-_y \psi^+_y + \bar\Psi^+_x \psi^+_x \bar\Psi^-_y \psi^-_y \\
& + \bar\Psi^+_x \psi^-_x \bar\Psi^+_y \psi^+_y + \bar\Psi^+_x \psi^-_x \bar\Psi^+_y \psi^-_y + \bar\Psi^+_x \psi^-_x \bar\Psi^-_y \psi^+_y + \bar\Psi^+_x \psi^-_x \bar\Psi^-_y \psi^-_y \\
& + \bar\Psi^-_x \psi^+_x \bar\Psi^+_y \psi^+_y + \bar\Psi^-_x \psi^+_x \bar\Psi^+_y \psi^-_y + \bar\Psi^-_x \psi^+_x \bar\Psi^-_y \psi^+_y + \bar\Psi^-_x \psi^+_x \bar\Psi^-_y \psi^-_y \\
& + \bar\Psi^-_x \psi^-_x \bar\Psi^+_y \psi^+_y + \bar\Psi^-_x \psi^-_x \bar\Psi^+_y \psi^-_y + \bar\Psi^-_x \psi^-_x \bar\Psi^-_y \psi^+_y + \bar\Psi^-_x \psi^-_x \bar\Psi^-_y \psi^-_y : \\
= & (-1)^4 \bar\Psi^-_y \psi^-_y \bar\Psi^+_x \psi^+_x + (-1)^3 \psi^-_x \psi^-_y \bar\Psi^+_x \bar\Psi^+_y + (-1)^2 \psi^-_x \bar\Psi^-_y \bar\Psi^+_x \psi^+_y \\
& + (-1)^2 \bar\Psi^-_x \psi^-_y \psi^+_x \bar\Psi^+_y + (-1) \bar\Psi^-_x \bar\Psi^-_y \psi^+_x \psi^+_y + \bar\Psi^-_x \psi^-_x \bar\Psi^+_y \psi^+_y \\
& + \text{'terms not contributing'}
\end{aligned}
$$

Of these six normal-ordered terms above only one represents a possible scattering process involving only fermions (as opposed to antifermions), namely those not containing any ψ^- and $\bar\Psi^+$ since those contain the b-modes for whom a pure fermion state is indistinguishable from the vacuum.

$$\psi(x) = \psi^+(x) + \psi^-(x) = \sum_s \left(\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a(p)\, u_s(p)\, e^{-ipx} + \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} b^+(p)\, v_s(p)\, e^{ipx} \right)$$

$$\bar\Psi(x) = \bar\Psi^-(x) + \bar\Psi^+(x) = \sum_s \left(\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a^+(p)\, \bar u_s(p)\, e^{ipx} + \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} b(p)\, \bar v_s(p)\, e^{-ipx} \right)$$

$\psi^- |p_1, \dots, p_n\rangle = \langle p_1, \dots, p_n| \bar\Psi^+ = 0$, where the p_i are an arbitrary number of fermions

Inserting this one relevant term for the time-ordered product in eq. (2)

$$S = -\tfrac{1}{2}(-ig)^2 \int d^4x \int d^4y\, D_F(x-y) \langle p, p' | \bar\Psi^-_x \bar\Psi^-_y \psi^+_x \psi^+_y | k, k' \rangle, \quad \text{where } |p, p'\rangle = \sqrt{2E_p}\sqrt{2E_{p'}}\, a^+_s(p)\, a^+_s(p')\, |0\rangle$$

We would like to commute the fields through the modes in the above expression. To do so, we calculate their commutation relations

$$\{\psi^+(x), a^+_s(p)\} = \sum_s \int \frac{d^3p'}{(2\pi)^3} \frac{e^{-ip'x}}{\sqrt{2E_{p'}}} u_s(p)\, [a_s(p'), a^+_s(p)] = \frac{e^{-ipx}}{\sqrt{2E_p}} u_s(p)$$

$$\{\bar\Psi^-(x), a_s(p)\} = \sum_{s'} \int \frac{d^3p'}{(2\pi)^3} \frac{e^{ip'x}}{\sqrt{2E_{p'}}} \bar u_{s'}(p')\, [a^+_{s'}(p'), a(p)] = \frac{e^{ipx}}{\sqrt{2E_p}} \bar u_s(p)$$

With these we can simplify $\psi^+_x \psi^+_y |p, p'\rangle$ to

$$\psi^+_x \psi^+_y \sqrt{2E_k}\sqrt{2E_{k'}}\, a^+_s(k)\, a^+_s(k')|0\rangle = -\sqrt{2E_k}\sqrt{2E_{k'}}\, \psi^+_x\, a^+_s(k)\, \psi^+_y\, a^+_s(k')|0\rangle + \sqrt{2E_{k'}}\, e^{-iky} u_s(k)\, \psi^+_x\, a^+_s(k')|0\rangle$$

$$= \sqrt{2E_k}\sqrt{2E_{k'}}\,\psi_x^+ a_s^+(k)\, a_s^+(k')\,\underbrace{\psi_y^+|0\rangle}_{0} - \sqrt{2E_k}\,e^{-ik'y}u_s(k')\,\psi_x^+\,a_s^+(k)\,|0\rangle - \sqrt{2E_k}\,e^{-iky}u_s(k)\,a_s^+(k')\underbrace{\psi_x^+|0\rangle}_{0}$$

$$+ e^{-iky}e^{-ik'x}u_s(k)u_s(k')|0\rangle = \sqrt{2E_k}\,e^{-iky}u_s(k)\,a_s^+(k)\underbrace{\psi_x^+|0\rangle}_{0} - e^{-ik'y}e^{-ikx}u_s(k)u_s(k)|0\rangle$$

$$+ e^{-iky}e^{-ik'x}u_s(k)u_s(k')|0\rangle = \left(e^{-ik'y}e^{-ikx}u_s(k)u_s(k') - e^{-iky}e^{-ik'x}u_s(k)u_s(k')\right)|0\rangle$$

$$\langle 0|\,a_s(p)\,a_s(p')\sqrt{2E_p}\sqrt{2E_{p'}}\,\bar\psi_x^-\,\bar\psi_y^- = -\langle 0|\,a_s(p)\,\bar\psi_x^-\,a_s(p')\,\bar\psi_y\sqrt{2E_p}\sqrt{2E_{p'}}$$

$$+ \langle 0|\,a_s(p)\,\bar\psi_y^-\sqrt{2E_p}\,e^{ip'x}\bar u_s(p') = -\langle 0|\,a_s(p)\,\bar\psi_y^-\sqrt{2E_{p'}}\,e^{ip'x}\bar u_s(p)$$

$$+ \langle 0|\,e^{ip'x}e^{ipy}\,\bar u_s(p)\,\bar u_s(p') = -\langle 0|\,e^{ipx}e^{ip'y}\,\bar u_s(p)\,\bar u_s(p) + \langle 0|\,e^{ip'x}e^{ipy}\,\bar u_s(p)\,\bar u_s(p')$$

$$= \langle 0|\left(e^{ip'x}e^{ipy}\,\bar u_s(p)\,\bar u_{s'}(p') - e^{ipx}e^{ip'y}\,\bar u_{s'}(p')\,\bar u_s(p)\right)$$

$$\langle p,p'|\,\bar\psi_x\bar\psi_y^-\,\psi_x^+\psi_y^+\,|k,k'\rangle = \langle 0|\,e^{i(p'-k)x}e^{i(p-k')y}\,\bar u_s(p)\,\bar u_s(p')\,u_r(k)\,u_r(k')\,|0\rangle$$

$$- \langle 0|\,e^{i(p'-k')x}e^{i(p-k)y}\,\bar u_s(p)\,\bar u_s(p')\,u_r(k)\,u_r(k) - \langle 0|\,e^{i(p-k)x}e^{i(p'-k')y}\,\bar u_s(p')\,\bar u_s(p)\,u_r(k)\,u_r(k')$$

$$+ \langle 0|\,e^{i(p-k')x}e^{i(p'-k)y}\,\bar u_s(p')\,\bar u_s(p)\,u_r(k')\,u_r(k)\,|0\rangle$$

Inserting $D_F^\phi(x-y) = \int \frac{d^4q}{(2\pi)^4}\frac{i\,e^{-iq(x-y)}}{q^2-m_\phi^2}$ into S, we can perform all integrals explicitly

$$S = -\frac{1}{2}(-ig)^2\int\frac{d^4q}{(2\pi)^4}\frac{i}{q^2-m_\phi^2}\int d^4x\int d^4y\Big(e^{i(p'-k-q)x}e^{i(p-k'+q)y}\,\bar u_s(p)\,\bar u_s(p')\,u_r(k)\,u_r(k')$$

$$- e^{i(p'-k'-q)x}e^{i(p-k+q)x}\,\bar u_s(p)\,\bar u_s(p')\,u_r(k')\,u_r(k) - e^{i(p-k-q)x}e^{i(p'-k'+q)y}\,\bar u_s(p')\,\bar u_s(p)\,u_r(k)\,u_r(k')$$

$$+ e^{i(p-k'-q)x}e^{i(p'-k+q)y}\,\bar u_s(p')\,\bar u_s(p)\,u_r(k')\,u_r(k)\Big)$$

$$= -\frac{1}{2}(-ig)^2\Big(\bar u_s(p)\,u_{r'}(k')\,\frac{i}{(p'-k)^2-m_\phi^2}\,\bar u_s(p')\,u_r(k)\,(2\pi)^4\delta^{(4)}(p-k'+p'-k)$$

$$- \bar u_s(p)\,u_r(k)\,\frac{i}{(p'-k')^2-m_\phi^2}\,\bar u_s(p')\,u_r(k')(2\pi)^4\delta^{(4)}(p-k+p'-k')$$

$$- \bar u_s(p)\,u_r(k)\,\frac{i}{(p-k)^2-m_\phi^2}\,\bar u_s(p')\,u_r(k')\quad(2\pi)^4\delta^{(4)}(p'-k'+p-k)$$

$$+ \bar u_s(p)\,u_r(k')\,\frac{i}{(p-k')^2-m_\phi^2}\,\bar u_s(p')\,u_r(k)\,(2\pi)^4\delta^{(4)}(p'-k+p-k')\Big)$$

$$\overset{p+p'=k+k'}{=} (-ig)^2\left(\bar u_s(p)\,u_r(k)\,\frac{i}{(p-k)^2-m_\phi^2}\,\bar u_s(p')\,u_r(k') - \bar u_s(p)\,u_r(k')\,\frac{i}{(p-k')^2-m_\phi^2}\,\bar u_s(p')\,u_r(k)\right)(2\pi)^4\delta^{(4)}(p+p'-k-k')$$

c) Based on this computation, guess the Feynman rules for the Yukawa interaction $-g\phi\bar{\Psi}\Psi$.

From the calculation performed in part b), we can guess the following Feynman rules:

- for every ingoing fermion Ψ of momentum p and spin s, associate a factor of $u_s(p) \hat{=} \xrightarrow{p,s}$
- for every outgoing fermion $\bar{\Psi}$ of momentum k and spin $-r$, associate a factor of $\bar{u}_r(k) \hat{=} \xrightarrow{k,r}$
- for every internal scalar ϕ of momentum q, associate a factor of $\frac{i}{q^2 - m_\phi^2} \hat{=} \text{--}q\text{--}$
- for every vertex $-g\phi\bar{\Psi}\Psi$, associate a factor of $-ig(2\pi)^4\delta^{(4)}(p+k-q)$ $\hat{=}$
- integrate over undetermined internal momenta $\int d^4 q_i$
- sum up all distinct Feynman diagrams to given order (with a relative for every two fermions crossing)

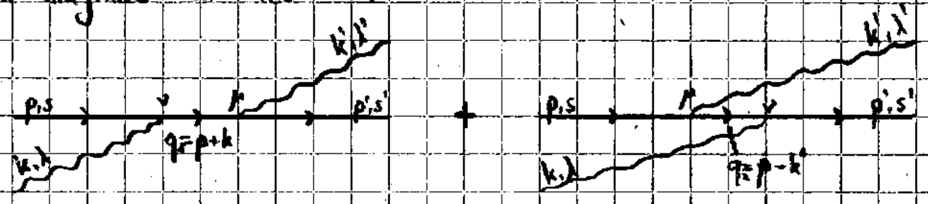
Problem 11.2 (The Klein-Nishina formula

We now look at the process of Compton scattering at tree-level in QED.

a) Use the Feynman rules in the script to write down the tree-level Compton scattering amplitude and bring it into the form

$$iM = ie^2 \epsilon_\mu^*(k',\lambda')\epsilon_\nu(k,\lambda)\bar{u}_{s'}(p')\left(\frac{\gamma^\mu \not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2pk} + \frac{\gamma^\nu \not{k}'\gamma^\mu - 2\gamma^\nu p^\mu}{2pk'}\right)u_s(p)$$

Compton scattering has contributions from exactly two distinct fully connected, amputated diagrams at tree-level

$iM =$ 

$$= \bar{u}_{s'}(p')(-ie\gamma^\mu)\epsilon_\mu^*(k',\lambda')\frac{i(\not{q}_1 + m_\psi)}{q_1^2 - m_\psi^2}\epsilon_\nu(k,\lambda)(-ie\gamma^\nu)u_s(p)$$

$$+ \bar{u}_{s'}(p')(-ie\gamma^\nu)\epsilon_\nu(k,\lambda)\frac{i(\not{q}_2 + m_\psi)}{q_2^2 - m_\psi^2}\epsilon_\mu^*(k',\lambda')(-ie\gamma^\mu)u_s(p)$$

$$= -ie^2\epsilon_\mu^*(k',\lambda')\epsilon_\nu(k,\lambda)\bar{u}_{s'}(p')\left(\gamma^\mu\frac{\not{p}+\not{k}+m_\psi}{(p+k)^2 - m_\psi^2}\gamma^\nu + \gamma^\nu\frac{\not{p}-\not{k}'+m_\psi}{(p-k')^2 - m_\psi^2}\gamma^\mu\right)u_s(p)$$

Using the energy-momentum relation of fermions and photons, we can simplify the denominators to $(p+k)^2 - m_\psi^2 = p^2 + 2pk + \underbrace{k^2}_{0} - m_\psi^2 = 2pk$ and $(p-k')^2 - m_\psi^2 = -2pk'$.

Simplification of the numerators requires γ-matrix manipulations and the Dirac eq.

$$(\not p + m_\psi)\gamma^\nu u_s(p) = (-\gamma^\nu \not p + 2\eta^{\mu\nu} p_\mu + m_\psi \gamma^\nu) u_s(p) = \gamma^\nu \underbrace{(-\not p + m_\psi) u_s(p)}_{0} + 2p^\nu u_s(p)$$

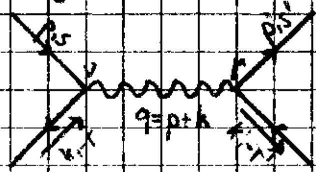
$$iM = -ie^2 \epsilon^*_\mu(k', \lambda') \epsilon_\nu(k, \lambda) \bar{u}_{s'}(p') \left(\frac{\gamma^\mu \not k \gamma^\nu + 2\gamma^\mu p^\nu}{2pk} + \frac{\gamma^\nu \not k' \gamma^\mu - 2\gamma^\nu p^\mu}{2pk'} \right) u_s(p)$$

b) Parts b) through g) of this exercise are not relevant for the exam.

<u>Imagined Exercise 1</u> (Electron-positron-to-muon-antimuon scattering)

Apply the momentum space Feynman rules of quantum electrodynamics to derive an algebraic expression for the full leading-order scattering amplitude of the process $e^+ + e^- \to \mu^+ + \mu^-$.

Electron-positron-to-muon-antimuon scattering has one diagram contributing at tree-level (leading order). The scattering amplitude therefore reads

$iM =$ 

$$= \bar{u}_{s'}(p')(-ie\gamma^\mu) v_{r'}(k') \frac{-i\eta_{\mu\nu}}{q^2} \bar{v}_r(k)(-ie\gamma^\nu) u_s(p)$$

$$= ie^2 \bar{u}_{s'}(p') \gamma^\mu v_{r'}(k') \frac{1}{(p+k)^2} \bar{v}_r(k) \gamma_\mu u_s(p) = \frac{i}{2} e^2 \bar{u}_{s'}(p') \gamma^\mu v_{r'}(k') \frac{1}{m^2 + pk} \bar{v}_r(k) \gamma_\mu u_s(p)$$

<u>Problem 9.3</u> (Trace identities)

Using just the algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \mathbb{1}_4$ (i.e. without resorting to a particular representation) and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ as defined above with $\not p = p_\mu \gamma^\mu$ and $S^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$, prove the following results:

Hint: It's helpful to note the cyclic properties of the trace, i.e. $Tr(ABC) = Tr(BCA)$, and you may find inserting $(\gamma^5)^2 = \mathbb{1}_4$ into a trace quite useful in some cases.

a) $Tr(\gamma^\mu) = 0$

$$Tr(\gamma^\mu) = Tr(\gamma^\mu (\gamma^5)^2) = Tr(\gamma^\mu \gamma^5 \gamma^5) = Tr(\gamma^5 \gamma^\mu \gamma^5) = -Tr(\gamma^\mu \gamma^5 \gamma^5) \implies Tr(\gamma^\mu) = 0$$

b) $Tr(\gamma^\mu \gamma^\nu) = 4\eta^{\mu\nu}$

$$Tr(\gamma^\mu \gamma^\nu) = Tr(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu - \gamma^\nu \gamma^\mu) = \underbrace{Tr(2\eta^{\mu\nu} \mathbb{1}_4)}_{8\eta^{\mu\nu}} - Tr(\gamma^\mu \gamma^\nu) \implies Tr(\gamma^\mu \gamma^\nu) = 4\eta^{\mu\nu}$$

c) $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda) = 0$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda) = \mathrm{Tr}((\gamma^5)^2\gamma^\mu\gamma^\nu\gamma^\lambda) = \mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^5) = (-1)^3\,\mathrm{Tr}(\gamma^5\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda) \Rightarrow \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda) = 0$$

d) $(\gamma^5)^2 = \mathbb{1}_4$

$$(\gamma^5)^2 = i\gamma^0\gamma^1\gamma^2\gamma^3\, i\gamma^0\gamma^1\gamma^2\gamma^3 = -(-1)^3\underbrace{(\gamma^0)^2}_{\mathbb{1}_4}\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 = (-1)^2\underbrace{(\gamma^1)^2}_{-\mathbb{1}_4}\gamma^2\gamma^3\gamma^2\gamma^3 = -(-1)\underbrace{(\gamma^2)^2}_{-\mathbb{1}_4}\underbrace{(\gamma^3)^2}_{-\mathbb{1}_4} = \mathbb{1}_4$$

e) $\mathrm{Tr}(\gamma^5) = 0$

$$\mathrm{Tr}(\gamma^5) = \mathrm{Tr}(i\gamma^0\gamma^1\gamma^2\gamma^3) = \mathrm{Tr}(i\gamma^1\gamma^2\gamma^3\gamma^0) = (-1)^3\,\mathrm{Tr}(i\gamma^0\gamma^1\gamma^2\gamma^3) = -\mathrm{Tr}(\gamma^5) \Rightarrow \mathrm{Tr}(\gamma^5) = 0$$

f) $\not{p}\not{q} = 2pq - \not{q}\not{p} = pq + 2S^{\mu\nu}p_\mu q_\nu$

$$\not{p}\not{q} = p_\mu\gamma^\mu q_\nu\gamma^\nu = p_\mu q_\nu\gamma^\mu\gamma^\nu = p_\mu q_\nu(-\gamma^\nu\gamma^\mu + 2\eta^{\mu\nu}\mathbb{1}_4) = 2p_\mu q^\mu\mathbb{1}_4 - q_\nu\gamma^\nu p_\mu\gamma^\mu$$

$$= 2pq\,\mathbb{1}_4 - \not{q}\not{p}$$

$$= p_\mu q_\nu\left(\gamma^\mu\gamma^\nu - \tfrac{1}{2}\gamma^\nu\gamma^\mu + \tfrac{1}{2}\gamma^\nu\gamma^\mu\right) = p_\mu q_\nu\left(2S^{\mu\nu} + \tfrac{1}{2}\underbrace{(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)}_{2\eta^{\mu\nu}\mathbb{1}_4}\right) = pq + 2S^{\mu\nu}p_\mu q_\nu$$

g) $\mathrm{Tr}(\not{p}\not{q}) = 4pq$

$$\mathrm{Tr}(\not{p}\not{q}) = \mathrm{Tr}(p_\mu\gamma^\mu q_\nu\gamma^\nu) = p_\mu q_\nu\,\mathrm{Tr}(\gamma^\mu\gamma^\nu) \overset{\text{part b)}}{=} p_\mu q_\nu\, 4\eta^{\mu\nu} = 4pq$$

h) $\mathrm{Tr}(\not{p}_1\ldots\not{p}_n) = 0$ if n is odd

$$\mathrm{Tr}(\not{p}_1\ldots\not{p}_n) = \mathrm{Tr}((\gamma^5)^2\not{p}_1\ldots\not{p}_n) = \mathrm{Tr}(\gamma^5\not{p}_1\ldots\not{p}_n\gamma^5) = (-1)^n\,\mathrm{Tr}((\gamma^5)^2\not{p}_1\ldots\not{p}_n) \Rightarrow 0 \text{ if } n \text{ is odd}$$

i) $\mathrm{Tr}(\not{p}_1\not{p}_2\not{p}_3\not{p}_4) = 4\left[(p_1p_2)(p_3p_4) + (p_1p_4)(p_2p_3) - (p_1p_3)(p_2p_4)\right]$

$$\mathrm{Tr}(\not{p}_1\not{p}_2\not{p}_3\not{p}_4) = p^1_\alpha p^2_\beta p^3_\gamma p^4_\delta\,\mathrm{Tr}(\gamma^\alpha\gamma^\beta(-\gamma^\delta\gamma^\gamma + 2\eta^{\gamma\delta}\mathbb{1}_4)) = p^1_\alpha p^2_\beta p^3_\gamma p^4_\delta\big(2\eta^{\gamma\delta}\,\mathrm{Tr}(\gamma^\alpha\gamma^\beta\mathbb{1}_4)$$

$$- \mathrm{Tr}(\gamma^\alpha\gamma^\beta\gamma^\delta\gamma^\gamma)\big) = p^1_\alpha p^2_\beta p^3_\gamma p^4_\delta\left(8\eta^{\gamma\delta}\eta^{\alpha\beta} - \mathrm{Tr}(\gamma^\alpha(-\gamma^\delta\gamma^\beta + 2\eta^{\beta\delta})\gamma^\gamma)\right)$$

$$= p^1_\alpha p^2_\beta p^3_\gamma p^4_\delta\left(8\eta^{\gamma\delta}\eta^{\alpha\beta} + \mathrm{Tr}(\gamma^\alpha\gamma^\delta\gamma^\beta\gamma^\gamma) - 2\eta^{\beta\delta}\,\mathrm{Tr}(\gamma^\alpha\gamma^\gamma)\right)$$

$$= p^1_\alpha p^2_\beta p^3_\gamma p^4_\delta\left(8\eta^{\gamma\delta}\eta^{\alpha\beta} - 8\eta^{\beta\delta}\eta^{\alpha\gamma} + \mathrm{Tr}((-\gamma^\delta\gamma^\alpha + 2\eta^{\alpha\delta})\gamma^\beta\gamma^\gamma)\right)$$

$$= p^1_\alpha p^2_\beta p^3_\gamma p^4_\delta \left(8\eta^{\gamma\delta}\eta^{\alpha\beta} - 8\eta^{\beta\delta}\eta^{\alpha\gamma} - \mathrm{Tr}(\gamma^\delta\gamma^\alpha\gamma^\beta\gamma^\gamma) + 2\eta^{\beta\delta}\,\mathrm{Tr}(\gamma^\alpha\gamma^\gamma)\right)$$

$$= 8(p_1\cdot p_2)(p_3\cdot p_4) - 8(p_1\cdot p_3)(p_2\cdot p_4) + 8(p_1\cdot p_4)(p_2\cdot p_3) - \mathrm{Tr}(\not p_4 \not p_1 \not p_2 \not p_3)$$

$$= 8\left((p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) - (p_1\cdot p_3)(p_2\cdot p_4)\right) - \mathrm{Tr}(\not p_4 \not p_1 \not p_2 \not p_3)$$

j) Parts j) through n) are skipped due to negligible relevance w.r.t. the exam.

Problem 11.1 (The Yukawa potential)

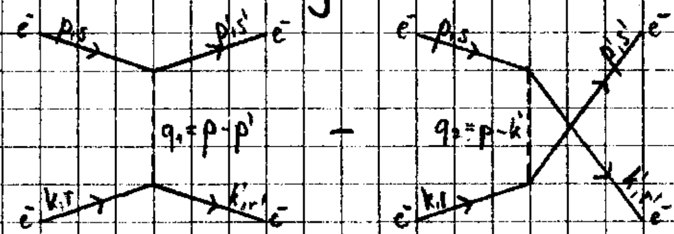
Here, we'll analyse and 'derive' the classical potential for Yukawa theory, where the interactions between fermions are transmitted by a scalar field ϕ. As discussed on assignment 10 in problem 10.2, the Lagrangian of this theory is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m_\phi^2\phi^2 + \overline{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi - g\phi\overline{\Psi}\Psi.$$

Analyse the following processes using the Feynman rules for Yukawa theory

a) electron-electron scattering: $e^- + e^- \to e^- + e^-$.

$i\mathcal{M} =$

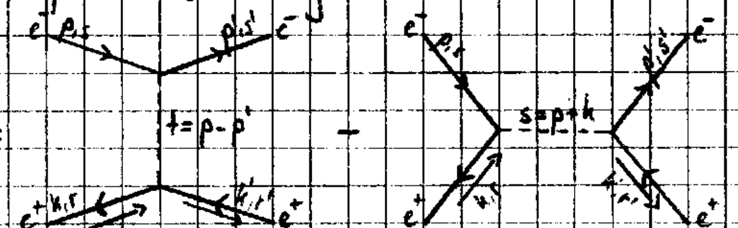


$$= \bar u_{s'}(p')(-ig)u_s(p)\frac{i}{q_1^2-m_\phi^2}\bar u_{r'}(k')(-ig)u_r(k) - \bar u_{r'}(k')(-ig)u_s(p)\frac{i}{q_2^2-m_\phi^2}\bar u_{s'}(p')(-ig)u_r(k)$$

$$= -ig^2\left(\bar u_{s'}(p')u_s(p)\frac{1}{(p-p')^2-m_\phi^2}\bar u_{r'}(k')u_r(k) - \bar u_{r'}(k')u_s(p)\frac{1}{(p-k')^2-m_\phi^2}\bar u_{s'}(p')u_r(k)\right)$$

b) electron-positron scattering: $e^+ + e^- \to e^+ + e^-$

$i\mathcal{M} =$



$$= \bar u_{s'}(p')(-ig)u_s(p)\frac{i}{t^2-m_\phi^2}\bar v_r(k)(-ig)v_{r'}(k') - \bar v_r(k)(-ig)u_s(p)\frac{i}{s^2-m_\phi^2}\bar u_{s'}(p')(-ig)v_{r'}(k')$$

$$= -ig^2\left(\bar u_{s'}(p')u_s(p)\frac{1}{(p-p')^2-m_\phi^2}\bar v_r(k)v_{r'}(k') - \bar v_r(k)u_s(p)\frac{1}{(p+k)^2-m_\phi^2}\bar u_{s'}(p')v_{r'}(k')\right)$$

Comparing your result to that of the Born approximation to the scattering amplitudes in nonrelativistic quantum mechanics, deduce the form and the sign of the classical Yukawa potential.

As part of the Born approximation, scattering electrons are regarded as distinguishable particles. Thus, of the two tree-level diagrams shown in part a) only one survives since with distinguishable particles, we can uniquely identify every outgoing particle with its ingoing counterpart. We consider the first one.

To evaluate its amplitude in the nonrelativistic limit, we keep only terms to lowest possible order in the 3-momenta, which is the second here, $\sigma(\vec{p}^2, \vec{p}'^2, \vec{k}^2, \vec{k}'^2)$, where

$$p = (m, \vec{p}), \qquad p' = (m, \vec{p}'), \qquad k = (m, \vec{k}), \qquad k' = (m, \vec{k}')$$

Therefore $(p-p')^2 = (m-m)^2 - (\vec{p}-\vec{p}')^2 = -(\vec{p}-\vec{p}')^2$ and $u_s(p) = \sqrt{m}\begin{pmatrix}\xi_s\\ \xi_s\end{pmatrix}$, $u_{s'}(p') = \sqrt{m}\begin{pmatrix}\xi_{s'}\\ \xi_{s'}\end{pmatrix}$, etc., where ξ_s is a two-component constant spinor normalized to $\xi_s \xi_{s'} = \delta_{ss'}$. The spinor products in the result of part a) are then

$$\bar{u}_{s'}(p') u_s(p) = \sqrt{m}\,(\xi_{s'}, \xi_{s'})\,\gamma^0 \sqrt{m}\begin{pmatrix}\xi_s\\ \xi_s\end{pmatrix} = 2m\, \xi_{s'} \xi_s = 2m\, \delta_{ss'}$$

$$\bar{u}_{r'}(k') u_r(k) = \sqrt{m}\,(\xi_{r'}, \xi_{r'})\,\gamma^0 \sqrt{m}\begin{pmatrix}\xi_r\\ \xi_r\end{pmatrix} = 2m\, \xi_{r'} \xi_r = 2m\, \delta_{r'r}$$

From the above, we can infer that the spin of each particle has to be separately conserved. With this result, we can rewrite the scattering amplitude as

$$iM = -ig^2\, 2m\, \delta_{ss'} \frac{1}{-(\vec{p}-\vec{p}')^2 - m_\phi^2}\, 2m\, \delta_{r'r} = \frac{ig^2}{(\vec{p}-\vec{p}')^2 - m_\phi^2}\, 2m\, \delta_{ss'}\, \delta_{r'r}$$

This can be compared with an analogous expression obtained by applying the Born approximation to solve this nonrelativistic scattering interaction:

$$\langle p' | iT | p \rangle = -i\tilde{V}(\vec{q})\,(2\pi)\,\delta(E_{p'} - E_p), \qquad \vec{q} = \vec{p}' - \vec{p}.$$

When making this comparison, we should be mindful of the different normalization of momentum eigenstates employed in relativistic QFT and nonrelativistic

QM, which gives us excess factors of $2m$ in the scattering amplitude. Also, the surplus factor of $(2\pi)^3\delta^{(3)}(\vec{p}-\vec{p}')$ that appears in the upper expression for the scattering amplitude when elevating it to the full S-matrix element disappears when performing the integral over the momentum of the target. Therefore, we can learn from our comparison that

$$\hat{V}(\vec{q}) = \frac{-g^2}{\vec{q}^2 + m_\phi^2}.$$

To find an expression for the potential $V(x)$ in position space, we Fourier transform

$$V(x) = \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\vec{x}}\, \hat{V}(\vec{q}) = \int \frac{d^3q}{(2\pi)^3} \frac{-g^2 e^{i\vec{q}\vec{x}}}{\vec{q}^2 + m_\phi^2} = -\frac{g^2}{(2\pi)^3} \int_0^\infty dq \int_0^\pi d\theta \int_0^{2\pi} d\varphi \, \frac{q^2 \sin\theta}{q^2+m_\phi^2}\, e^{iqr\cos\theta}, \quad \begin{aligned} y &= \cos\theta \\ dy &= -\sin\theta\, d\theta \end{aligned}$$

$$= \frac{g^2}{4\pi^2} \int_0^\infty dq \int_1^{-1} \frac{q^2}{q^2+m_\phi^2}\, e^{iqry}\, dy = -\frac{g^2}{4\pi^2} \int_0^\infty dq \frac{q^2}{q^2+m_\phi^2} \frac{1}{iqr}\left(e^{iqr} - e^{-iqr}\right)$$

$$= -\frac{g^2}{4\pi^2 ir}\left(\int_0^\infty dq \frac{q}{q^2+m_\phi^2} e^{iqr} - \int_0^{-\infty} (-dq) \frac{-q}{q^2+m_\phi^2}\, e^{iqr}\right) = -\frac{g^2}{4\pi^2 ir} \int_{-\infty}^\infty dq \underbrace{\frac{q}{q^2+m_\phi^2}\, e^{iqr}}_{f(q)}$$

This can be evaluated as a complex contour integral by closing it in the upper half plane of $\mathbb{C}$, where we pick up the residue of the first order pole at $q = +im_\phi$ [since $q^2 + m_\phi^2 = (q-im_\phi)(q+im_\phi)$] which is given by

$$\mathrm{Res}\left(f(q), q = im_\phi\right) = \lim_{q\to im_\phi} (q-im_\phi) f(q) = \lim_{q\to im_\phi} \frac{q}{q+im_\phi}\, e^{iqr} = \frac{im_\phi}{2im_\phi}\, e^{-m_\phi r} = \frac{1}{2} e^{-m_\phi r}$$

Now applying the residue theorem, we get

$$V(x) = -\frac{g^2}{4\pi^2 ir}\, 2\pi i \sum_{q_0 \in \mathbb{C}_+} \mathrm{Res}\left(f(q), q = q_0\right) = -\frac{g^2}{4\pi^2 ir}\, 2\pi i\, \frac{1}{2} e^{-m_\phi r} = -\frac{g^2}{4\pi r} e^{-m_\phi r},$$

which is an attractive potential since it shrinks with decreasing r. The same result, i.e. an attractive force produced by the Yukawa potential, is obtained when considering scattering of antifermions and fermions with antifermions.

Problem 8.1 (Relativistic kinematics for 2-2 scattering)

Consider the S-matrix element for non-trivial scattering of two particles

$$\langle \vec{p}_3, \vec{p}_4 | S | \vec{p}_1, \vec{p}_2 \rangle = i(2\pi)^4 \delta^{(4)}(p_3 + p_4 - p_1 - p_2)\, \mathcal{M}(p_1, p_2, p_3, p_4).$$

a) Show that the total cross-section is

$$\sigma = \frac{1}{2!} \frac{1}{64\pi^2 s} \int d^2\Omega_3 |\mathcal{M}|^2,$$

or equivalently the differential cross-section $\frac{d^2\sigma}{d^2\Omega_3} = \frac{1}{2!} \frac{1}{64\pi^2 s} |\mathcal{M}|^2$. Here, we make use of the relativistically invariant Mandelstam variable $s = (p_1 + p_2)^2$ and $\int d^2\Omega_3$ refers to integration over $\vec{p}_3$ as $\int d^3p_3 = \int d|\vec{p}_3| \int d^2\Omega_3$.

In the rest frame of particle 2, it holds that

$$4|p_1^1| m\sigma = \frac{1}{2!} \int \frac{d^3p_3}{(2\pi)^3} \frac{1}{2E_3} \int \frac{d^3p_4}{(2\pi)^3} \frac{1}{2E_4} (2\pi)^4 \delta^{(4)}(p_3 + p_4 - p_1 - p_2) |\mathcal{M}(p_1, p_2, p_3, p_4)|^2$$

To perform the right-hand integral, we may move into the centre-of-mass system since all parts of the above expression are Lorentz-invariant and therefore valid in any frame of reference. In the CoM-frame, we have $p_1 + p_2 = p_3 + p_4 = (\sqrt{s}, 0, 0, 0)$ and hence $\vec{p}_1 + \vec{p}_2 = \vec{p}_3 + \vec{p}_4 = \vec{0}$. We can therefore write for the above.

$$4|p_1^1| m\sigma = \frac{1}{8(2\pi)^2} \int d^3p_3 \int d^3p_4 \frac{1}{E_3 E_4} \delta^{(1)}(E_3 + E_4 - E_1 - E_2)\, \delta^{(3)}(\vec{p}_3 + \vec{p}_4 \underbrace{- \vec{p}_1 - \vec{p}_2}_{\vec{0}}) |\mathcal{M}(p_1, p_2, p_3, p_4)|^2$$

$$= \frac{1}{32\pi^2} \int d^3p_3 \int d^3p_4 \frac{1}{E_3 E_4} \delta^{(1)}(E_3 + E_4 - \sqrt{s})\, \delta^{(3)}(\vec{p}_3 + \vec{p}_4) |\mathcal{M}(p_1, p_2, p_3, p_4)|^2$$

$$= \frac{1}{32\pi^2} \int d^3p_3 \frac{1}{E_3^2} \delta^{(1)}(2E_3 - \sqrt{s}) |\mathcal{M}(p_1, p_2, p_3)|^2, \quad \text{where } \int d^3p_3 = \int d^2\Omega_3 \int d|\vec{p}_3|.$$

Here, we can use that $\delta(f(x)) = \sum_{i=1}^{n} \frac{\delta(x - x_i)}{|f'(x_i)|}$ where i sums over the (by necessity) finite set of simple poles of $f(x)$. In this case

$$\delta^{(1)}(2E_3 - \sqrt{s}) = \delta^{(1)}\Big(\underbrace{2\sqrt{\vec{p}_3^2 + m^2}}_{f(|\vec{p}_3|)} - \sqrt{s(|\vec{p}_3|)}\Big) = \frac{1}{|f'(|\vec{p}_3|)|} \delta^{(1)}\left(f^{-1}(f(|\vec{p}_3|)) - f^{-1}(\sqrt{s(|\vec{p}_3|)})\right)$$

$$= \left(\frac{2|\vec{p}_3|}{\sqrt{\vec{p}_3^2 + m^2}}\right)^{-1} \delta^{(1)}\left(|\vec{p}_3| - \sqrt{\frac{s}{4} - m^2}\right)$$

Inserting this reformulation of the delta distribution into our integral gives

$$4|\vec{p}_1^{\,L}|\,\sigma m = \frac{1}{32\pi^2}\int d^2\Omega_3 \int d|\vec{p}_3|\,|\vec{p}_3|^2 \frac{1}{\vec{p}_3^{\,2}+m^2}\,\frac{\sqrt{\vec{p}_3^{\,2}+m^2}}{2|\vec{p}_3|}\,\delta\left(|\vec{p}_3|-\sqrt{\frac{s}{4}-m^2}\right)|M(p_1,p_2,p_3)|^2$$

$$= \frac{1}{64\pi^2}\int d^2\Omega_3\,\frac{\sqrt{\frac{s}{4}-m^2}}{\sqrt{\frac{s}{4}-m^2+m^2}}\,|M(p_1,p_2,\Omega_3)|^2 = \frac{1}{64\pi^2}\sqrt{\frac{s-4m^2}{s}}\int d^2\Omega_3\,|M(p_1,p_2,\Omega_3)|^2$$

In the lab frame (rest frame of particle 2), we have $p_2=(m,0,0,0)$, $p_1=(E_1^L,\vec{p}_1^{\,L})$

$$s=(p_1+p_2)^2 = p_1^2+2p_1p_2+p_2^2 = 2m^2+2mE_1^L \implies E_1^L=\frac{s-2m^2}{2m}$$

$$|\vec{p}_1^{\,L}| = \sqrt{(E_1^L)^2-m^2} = \sqrt{\frac{(s-2m^2)^2}{4m^2}-\frac{4m^4}{4m^2}} = \sqrt{\frac{s^2-4sm^2+4m^4-4m^4}{4m^2}} = \frac{\sqrt{s(s-4m^2)}}{2m}$$

And therefore, we can write

$$\sigma = \frac{1}{4|\vec{p}_1^{\,L}|m}\,\frac{1}{64\pi^2}\sqrt{\frac{s-4m^2}{s}}\int d^2\Omega_3\,|M(p_1,p_2,\Omega_3)|^2 = \frac{1}{2\sqrt{s(s-4m^2)}}\,\frac{1}{64\pi^2}\sqrt{\frac{s-4m^2}{s}}\int d^2\Omega_3\,|M(p_1,p_2,\Omega_3)|^2$$

$$= \frac{1}{128\pi^2 s}\int d^2\Omega_3\,|M(p_1,p_2,\Omega_3)|^2$$