Imperial College London

MSci EXAMINATION May 2009

This paper is also taken for the relevant Examination for the Associateship

QUANTUM FIELD THEORY

For 4th-Year Physics Students

Thursday, 21st May 2009: 14:00 to 16:00

Answer THREE questions.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

1. Consider the classical real scalar field $\phi(x)$ with Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi).$$

You are given that the classical Poisson bracket satisfies

$$\big\{\phi(t,\boldsymbol{x}),\pi(t,\boldsymbol{y})\big\}_{\text{PB}} = -\big\{\pi(t,\boldsymbol{y}),\phi(t,\boldsymbol{x})\big\}_{\text{PB}} = \delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}),$$

while $\big\{\phi(t,\boldsymbol{x}),\phi(t,\boldsymbol{y})\big\}_{\text{PB}} = 0$ and $\big\{AB,C\big\}_{\text{PB}} = A\big\{B,C\big\}_{\text{PB}} + B\big\{A,C\big\}_{\text{PB}}$.

(i) What is the definition of the momentum density $\pi(x)$ conjugate to $\phi(x)$? What is it equal to in this case?

Rewrite $\mathcal{L}$ in terms of $\dot\phi$ and $\boldsymbol{\nabla}\phi$ and hence show that the Hamiltonian H is given by

$$H = \int \mathrm{d}^3x \left(\tfrac{1}{2}\pi^2 + \tfrac{1}{2}\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi + V(\phi)\right).$$

What is $V(\phi)$ for a free scalar field of mass m? What is the minimum value of H in this case? [6 marks]

(ii) Define

$$\begin{aligned} T^{\mu\nu} &= \partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}\left(\tfrac{1}{2}\partial_\lambda\phi\partial^\lambda\phi - V(\phi)\right), \\ M^{\mu\nu\rho} &= T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu. \end{aligned}$$

Show that $\partial_\mu T^{\mu\nu} = 0$ if ϕ satisfies the equation of motion $\partial_\mu\partial^\mu\phi + V'(\phi) = 0$ and hence that $\partial_\mu M^{\mu\nu\rho} = 0$. [5 marks]

(iii) Consider the integrals

$$\begin{aligned} Q^\mu &= \int \mathrm{d}^3x T^{0\mu}(t,\boldsymbol{x}), \\ Q^{\mu\nu} &= \int \mathrm{d}^3x M^{0\mu\nu}(t,\boldsymbol{x}). \end{aligned}$$

How do they depend on t? What does Q^μ represent physically? What about Q^{ij}, where $i,j = 1,2,3$?

Write down the components of Q^μ and show explicitly that one is related to H. [4 marks]

(iv) Show that $\big\{Q^\mu,\phi(x)\big\}_{\text{PB}} = -\partial^\mu\phi(x)$ and comment on the result.

Comment briefly on what you expect for the Poisson brackets between different components of Q^μ and $Q^{\mu\nu}$. [5 marks]

[Total 20 marks]

2. Consider a free real scalar field $\phi(x)$ with conjugate momentum density $\pi(x) = \dot{\phi}(x)$. Define the operator

$$a_{\boldsymbol{p}} = \int \mathrm{d}^3x \frac{\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}}{\sqrt{2E_{\boldsymbol{p}}}} \left[E_{\boldsymbol{p}}\phi(0,\boldsymbol{x}) + \mathrm{i}\pi(0,\boldsymbol{x})\right],$$

where $E_{\boldsymbol{p}} = \sqrt{|\boldsymbol{p}|^2 + m^2}$.

(i) The equal-time commutation relations (ETCRs) state that

$$\left[\phi(t,\boldsymbol{x}), \pi(t,\boldsymbol{y})\right] = \mathrm{i}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}).$$

Are the fields $\phi(x)$ and $\pi(x)$ in the Schrödinger or Heisenberg picture? Why is this picture more natural in a relativistic theory?

What are the ETCRs for $\left[\phi(t,\boldsymbol{x}), \phi(t,\boldsymbol{y})\right]$ and $\left[\pi(t,\boldsymbol{x}), \pi(t,\boldsymbol{y})\right]$? [4 marks]

(ii) Give an expression for $a^\dagger_{\boldsymbol{p}}$. Using the ETCRs show that

$$\left[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}\right] = (2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}),$$
$$\left[a_{\boldsymbol{p}}, a_{\boldsymbol{q}}\right] = 0, \qquad \left[a^\dagger_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}\right] = 0.$$

[6 marks]

(iii) Given

$$\phi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \left(a_{\boldsymbol{p}}\mathrm{e}^{-\mathrm{i}p\cdot x} + a^\dagger_{\boldsymbol{p}}\mathrm{e}^{\mathrm{i}p\cdot x}\right),$$

where $p^0 = E_{\boldsymbol{p}}$, use the results from part ii to show that at unequal times

$$\left[\phi(x), \phi(y)\right] = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} \left(\mathrm{e}^{-\mathrm{i}p\cdot(x-y)} - \mathrm{e}^{\mathrm{i}p\cdot(x-y)}\right).$$

What properties must this commutator have if the field theory is to respect microcausality? [5 marks]

(iv) You are given that $\delta(x^2 - a^2) = (1/2a)\left[\delta(x-a) - \delta(x+a)\right]$. Show that the expression for $\left[\phi(x), \phi(y)\right]$ given in part iii can be rewritten as

$$\left[\phi(x), \phi(y)\right] = \int \frac{\mathrm{d}^4p}{(2\pi)^3} \delta(p^2 - m^2)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)},$$

where $\mathrm{d}^4p = \mathrm{d}^3p\,\mathrm{d}p^0$. Comment on the Lorentz transformation properties of this expression, and give a brief argument that the ETCRs take the same form in all inertial frames.

What is the significance of this result? [5 marks]

[Total 20 marks]

3. This question is about the free classical Dirac field $\psi(x)$. The Lagrangian is given by

$$\mathcal{L} = \mathrm{i}\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi,$$

where the gamma matrices γ^μ satisfy $\left\{\gamma^\mu, \gamma^\nu\right\} = 2\eta^{\mu\nu}\mathbb{1}$ and $\bar{\psi} = \psi^\dagger\gamma^0$.

(i) The field $\psi(x)$ is a spinor. How many components does it have? After quantization what is the spin of the corresponding single particle states?

Treating $\psi(x)$ and $\bar{\psi}(x)$ as independent fields, show that the Euler–Lagrange equation for $\bar{\psi}(x)$ is the Dirac equation

$$\mathrm{i}\gamma^\mu\partial_\mu\psi - m\psi = 0.$$

[4 marks]

(ii) Using $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$ show that if $\psi(x)$ satisfies the Dirac equation then

$$\mathrm{i}\partial_\mu\bar{\psi}(x)\gamma^\mu + m\bar{\psi}(x) = 0.$$

Hence show that the current $j^\mu = \bar{\psi}\gamma^\mu\psi$ is conserved, that is $\partial_\mu j^\mu = 0$, if ψ satisfies the Dirac equation. [4 marks]

(iii) Show that the complex conjugate of $\mathcal{L}$ is given by

$$\mathcal{L}^* = \mathcal{L} - \mathrm{i}\partial_\mu\left(\bar{\psi}\gamma^\mu\psi\right).$$

Consider the real Lagrangian

$$\mathcal{L}' = \tfrac{1}{2}\mathrm{i}\left[\bar{\psi}\gamma^\mu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma^\mu\psi\right] - m\bar{\psi}\psi.$$

Treating $\psi(x)$ and $\bar{\psi}(x)$ as independent fields, show that the corresponding Euler–Lagrange equation for $\bar{\psi}(x)$ is again the Dirac equation. [5 marks]

(iv) The matrix $\gamma_5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$ has the property that $\left\{\gamma_5, \gamma_\mu\right\} = 0$. Show that the "axial vector" current $j_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi$ satisfies

$$\partial_\mu j_5^\mu = 2\mathrm{i}m\bar{\psi}\gamma_5\psi.$$

What does this result imply about the symmetries of the Dirac equation when $m = 0$?

Identify the corresponding infinitesimal transformation of ψ and demonstrate directly whether or not $\mathcal{L}$ is invariant under this transformation. [7 marks]

[Total 20 marks]

4. Consider a free Dirac field $\psi(x)$ given by

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \sum_{s=1}^{2} \left[a^s_{\boldsymbol{p}} u^s(p) e^{-ip\cdot x} + b^{s\dagger}_{\boldsymbol{p}} v^s(p) e^{ip\cdot x} \right],$$

where $p^0 = E_{\boldsymbol{p}} = \sqrt{|\boldsymbol{p}|^2 + m^2}$. The non-vanishing anti-commutation relations are

$$\left\{ a^r_{\boldsymbol{p}}, a^{s\dagger}_{\boldsymbol{q}} \right\} = \left\{ b^r_{\boldsymbol{p}}, b^{s\dagger}_{\boldsymbol{q}} \right\} = (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{q}) \delta^{rs}.$$

The Hamiltonian H and the operator Q are given by

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_{s=1}^{2} E_{\boldsymbol{p}} \left(a^{s\dagger}_{\boldsymbol{p}} a^s_{\boldsymbol{p}} + b^{s\dagger}_{\boldsymbol{p}} b^s_{\boldsymbol{p}} \right),$$

$$Q = \int \frac{d^3p}{(2\pi)^3} \sum_{s=1}^{2} \left(a^{s\dagger}_{\boldsymbol{p}} a^s_{\boldsymbol{p}} - b^{s\dagger}_{\boldsymbol{p}} b^s_{\boldsymbol{p}} \right).$$

(i) Define the vacuum state $|0\rangle$, the single particle states $|\boldsymbol{p}, s, -\rangle$ and $|\boldsymbol{p}, s, +\rangle$. What do the labels $\boldsymbol{p}$, s and $\pm$ denote?
Define a two-particle state, and show that the particles are fermions.
[6 marks]

(ii) Show that

$$a^{r\dagger}_{\boldsymbol{q}} a^r_{\boldsymbol{q}} |\boldsymbol{p}, s, -\rangle = (2\pi)^3 \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}) \delta^{rs} \; |\boldsymbol{q}, r, -\rangle,$$
$$b^{r\dagger}_{\boldsymbol{q}} b^r_{\boldsymbol{q}} |\boldsymbol{p}, s, +\rangle = (2\pi)^3 \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}) \delta^{rs} \; |\boldsymbol{q}, r, +\rangle.$$

Hence show that $|0\rangle$, $|\boldsymbol{p}, s, -\rangle$ and $|\boldsymbol{p}, s, +\rangle$ are eigenstates of H and Q and give the eigenvalues. [7 marks]

(iii) Using the results of part ii write down an operator S_z that has eigenvalues

$$S_z|0\rangle = 0, \quad S_z|\boldsymbol{p}, 1, \pm\rangle = \pm\tfrac{1}{2}|\boldsymbol{p}, 1, \pm\rangle, \quad S_z|\boldsymbol{p}, 2, \pm\rangle = \mp\tfrac{1}{2}|\boldsymbol{p}, 2, \pm\rangle.$$

[2 marks]

(iv) One could also try to quantize the Dirac field by using commutation relations. Specifically the only non-vanishing commutators are then

$$\left[a^r_{\boldsymbol{p}}, a^{s\dagger}_{\boldsymbol{q}} \right] = -\left[b^r_{\boldsymbol{p}}, b^{s\dagger}_{\boldsymbol{q}} \right] = (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{q}) \delta^{rs}.$$

and the sign of the $b^{s\dagger}_{\boldsymbol{p}} b^s_{\boldsymbol{p}}$ term in the Hamiltonian H changes.
If we define states in the Hilbert space in the conventional way, what are the two problems with this quantization prescription?
How can one problem be alleviated by changing the definition of the Hilbert space? [5 marks]

[Total 20 marks]

5. In both the interaction picture and the Heisenberg picture operators evolve with time. One has

$$\partial_t \mathcal{O}_I = \mathrm{i}\big[H_{0,I}, \mathcal{O}_I\big], \qquad \partial_t \mathcal{O}_H = \mathrm{i}\big[H_H, \mathcal{O}_H\big],$$

where in the interaction picture, $\mathcal{O}_I$ is any operator and $H_{0,I}$ is the free Hamiltonian, while in the Heisenberg picture $\mathcal{O}_H$ is an operator and H_H is the total Hamiltonian.

(i) What are the corresponding expressions for the evolution of states $|\psi(t)\rangle_H$ and $|\psi(t)\rangle_I$ in the Heisenberg and interaction pictures?

Comment briefly on why the interaction picture is useful in perturbation theory.

Write down the full Lagrangian density for "phi-fourth" theory and identify $\mathcal{L}_0$ and $\mathcal{L}_{\text{int}}$, the free and interaction Lagrangian densities, that contribute to the free and interaction Hamiltonians respectively. [5 marks]

(ii) Show that in the two pictures $\partial_t H_H = 0$ and $\partial_t H_{0,I} = 0$. Hence show that

$$\mathcal{O}_H(t) = \mathrm{e}^{\mathrm{i}H_H(t-t_0)}\mathcal{O}_H(t_0)\,\mathrm{e}^{-\mathrm{i}H_H(t-t_0)},$$
$$\mathcal{O}_I(t) = \mathrm{e}^{\mathrm{i}H_{0,I}(t-t_0)}\mathcal{O}_I(t_0)\,\mathrm{e}^{-\mathrm{i}H_{0,I}(t-t_0)},$$

are solutions of the evolution equations for $\mathcal{O}_H(t)$ and $\mathcal{O}_I(t)$. [4 marks]

(iii) Let operators in the two pictures be related by

$$\mathcal{O}_I(t) = U(t, t_0)\mathcal{O}_H(t)U(t, t_0)^{-1},$$

where $U(t_0, t_0) = \mathbb{1}$.

Using the results of part ii write an expression for $U(t, t_0)$. Show that it satisfies

$$\partial_t U = -\mathrm{i}H_{\text{int},I}U,$$

where $H_{\text{int},I}$ is the interaction Hamiltonian in the interaction picture.

[4 marks]

(iv) Define the time-ordered product

$$T H_{\text{int},I}(t_1)H_{\text{int},I}(t_2).$$

As an expansion in $H_{\text{int},I}$, show explicitly that

$$U(t, t_0) = \mathbb{1} - \mathrm{i}\int_{t_0}^{t} \mathrm{d}t_1 H_{\text{int},I}(t_1) + \tfrac{1}{2}(-\mathrm{i})^2\int_{t_0}^{t}\int_{t_0}^{t} \mathrm{d}t_1\mathrm{d}t_2 T H_{\text{int},I}(t_1)H_{\text{int},I}(t_2) + \dots$$

is a solution for $U(t, t_0)$ to second order in $H_{\text{int},I}$. [7 marks]

[Total 20 marks]

6. This question is about Feynman diagrams in "phi-fourth" theory. Recall that in the interaction picture the S-matrix is given by

$$S = T \exp\left(\mathrm{i} \int \mathrm{d}^4x : \mathcal{L}_{\mathrm{int}}(x) :\right).$$

(i) Explain how S can be described as a perturbation expansion and write down the first three terms in the expansion.

The scattering amplitude $\mathrm{i}\mathcal{M}$ is usually defined to be

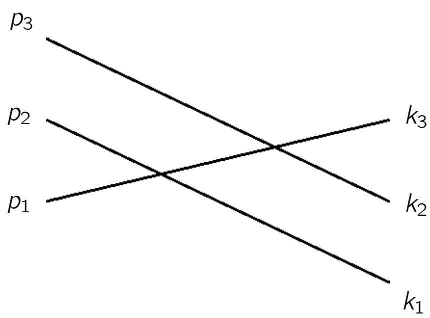
$$\langle \mathrm{out}|\mathrm{i}T|\mathrm{in}\rangle = \mathrm{i}\mathcal{M}\ (2\pi)^4 \delta^{(4)}\left(p_{\mathrm{out}} - p_{\mathrm{in}}\right)$$

where $S = \mathbb{1} + \mathrm{i}T$. Why is the $\mathbb{1}$ contribution not included? What are p_{out} and p_{in} and what is the physical meaning of the δ-function? [4 marks]

(ii) Consider the scattering of three incoming ϕ particles with momenta k_1, k_2 and k_3 to three outgoing particles with momenta p_1, p_2 and p_3.

Define the $|\mathrm{in}\rangle$ and $|\mathrm{out}\rangle$ states for this process.

Use the position-space Feynman rules to calculate the contribution of the following Feynman diagram to $\langle \mathrm{out}|\mathrm{i}T|\mathrm{in}\rangle$ in terms of the propagator $D_F(x-y)$.



Given

$$D_F(x-y) = \mathrm{i} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{p^2 - m^2 + \mathrm{i}\epsilon}$$

write this contribution as a function only of momenta.

Show that this agrees with the contribution to $\mathrm{i}\mathcal{M}$ calculated using the momentum space rules. [7 marks]

(iii) Draw a second Feynman diagram that has no loops, is not related to the diagram in part ii by a permutation of the p_i momenta or of the k_i momenta, and contributes to $\mathrm{i}\mathcal{M}$ at the same order in λ.

Evaluate this diagram using the momentum-space Feynman rules. [3 marks]

Now consider the scattering of two incoming ϕ particles with momenta q_1 and q_2 to two outgoing ϕ particles with momenta p_1 and p_2.

Taking the non-relativistic limit and comparing with the Born approximation gives

$$\mathrm{i}\mathcal{M}(q_1, q_2, p_1, p_2) = -\mathrm{i}\tfrac{1}{2}\left[\tilde{V}(\boldsymbol{p}_1 - \boldsymbol{q}_1) + \tilde{V}(\boldsymbol{p}_1 - \boldsymbol{q}_2)\right],$$

where $\tilde{V}(\boldsymbol{q})$ is the Fourier transform of the classical potential $V(\boldsymbol{x})$ between the two particles.

(iv) Use the momentum-space Feynman rules to identify $\tilde{V}(\boldsymbol{k})$ and hence calculate the form of $V(\boldsymbol{x})$ at order λ.

Draw a Feynman diagram that give corrections to $\mathrm{i}\mathcal{M}$ (and hence potentially to $V(\boldsymbol{x})$) at order λ^2. [6 marks]

[Total 20 marks]