

1. ABSTRACT GROUPS

Definition: A group G

$$G \times G \rightarrow G$$

$$(g_1, g_2) \mapsto g_1 g_2$$

s.t. (i) Associativity: $(g_1 g_2) g_3 = g_1 (g_2 g_3)$

(ii) Identity: $\exists e \in G: e \cdot g = g \quad \forall g \in G$

(iii) Inverse: $\forall g \in G \quad \exists g^{-1}: g^{-1} g = e$

In case: $\forall g, h \in G \quad gh = hg$, then G is Abelian (commutative)

Remark: (i) right-inverses

$$g^{-1} g = e$$

$$(g^{-1} g) g^{-1} = e g^{-1} = g^{-1}$$

$$\begin{matrix} \text{"} \\ g^{-1} (g g^{-1}) \end{matrix} \Rightarrow \boxed{g g^{-1} = e}$$

(ii) right-identity:

$$g \cdot e = g(g^{-1} g) = (g g^{-1}) g = e g = g$$

(iii) Uniqueness of e :

$$\text{Assume } \exists e': e' g = g \quad \forall g$$

$$e' \cdot e' = e = e'$$

(iv) uniqueness of inverse:

$$\text{Assume that for } g \in G \quad \exists h: hg = e$$

$$h = h \cdot e = h(g g^{-1}) = (hg) g^{-1} = e g^{-1} = g^{-1}$$

Examples:

(i) $(\mathbb{Z}, +)$ Abelian group

(ii) Cyclic group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$

(iii) Symmetric group S_n
(permutations of n objects)

(iv) Dihedral group D_3
(symmetries of a regular triangle Δ)

Example: Dihedral group D_3 :

• D_3

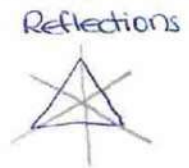


Elements (6):

$$|D_3| = 6 = \{ e_0 = R_0, R_1 = R(\frac{2\pi}{3}), R_2 = R(\frac{4\pi}{3}), \underbrace{S_0, S_1, S_2}_{\text{Reflections}} \}$$

$$\begin{aligned} R_i R_j &= R_{i+j \bmod 3} \\ S_i^2 &= R_0 \\ R_i S_j &= S_{i+j \bmod 3} \end{aligned}$$

✓ these properties define the group



Using this relations: ✓ these are consequences

$$S_i R_j = S_{i-j}$$

$$S_i S_j = R_{-i+j} \quad (\text{check})$$

• D_n : sym. of regular n -gon

"

$$\{ R_0, R_1, \dots, R_{n-1}, S_0, \dots, S_{n-1} \}$$

(*)

So far, all examples are discrete.

But now: Continuous ones.

Lie groups.

(v) translation group $(\mathbb{R}^n, +)$ Abelian.

(vi) General linear group:

$$GL(n) := \{ A \in \text{Mat}(n, n, \mathbb{R}) \mid \det(A) \neq 0 \}$$

(vii) (special) orthogonal group:

$$(S) O(n) = \{ A \in \text{Mat}(n, n, \mathbb{R}) \mid A^t A = 1_n \} \quad (\det(A) = 1)$$

(viii) Euclidean group (Sym. of a affine group)

$$E(n) := \{ (a, A) \in \mathbb{R}^n \times O(n) \}$$

$$(a, A)(a', A') = (a + Aa', AA')$$

(ix) (special) unitary group

$$(S) U(n) := \{ A \in \text{Mat}(n, n, \mathbb{C}) \mid \bar{A}^t = A^t A = 1_n \} \quad (\det(A) = 1)$$

Def: A group-homomorphism:

$$\psi: G \longrightarrow H \text{ s.t.}$$

$$\psi(g_1 g_2) = \psi(g_1) \psi(g_2)$$

• If ψ is invertible, group isomorphism

• $\text{Hom}(G, H)$ set of homomorphisms.

Rem: (i) $\psi(e_G) = e_H$

$$[\psi(g) = \psi(g \cdot e_G) = \psi(g) \cdot \psi(e_G) \Rightarrow \psi(e_G) = e_H]$$

(ii) $\psi(g^{-1}) = (\psi(g))^{-1}$

$$[\psi(g^{-1})\psi(g) = \psi(g^{-1}g) = \psi(e_G) = e_H \Rightarrow \psi(g^{-1}) = (\psi(g))^{-1}]$$

Examples:

(i) $\mathbb{Z}_n \longrightarrow D_n$

$$[a] \longmapsto R_{a \bmod n} \quad R_i \cdot R_j = R_{i+j}$$

(ii) $\mathbb{Z} \xrightarrow{n} U(1) = \{a \in \mathbb{C} \mid \bar{a}a = 1\}$

$$[a] \longmapsto e^{\frac{2\pi i a}{n}}$$

(iii) $\psi: O(2) \longrightarrow O(3)$

$$A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

(iv) $U(n) \longrightarrow U(1)$

$$A \longmapsto \det(A)$$



Def: A subgroup $H \subset G$, which is itself a group w.r.t. multiplication in G

Rem: (i) H has to contain $e_G = e_H$

(ii) $H \subset G \iff \forall h_1, h_2 \in H \quad h_1 h_2^{-1} \in H$

(iii) Trivial subgroups $\{e\} \subset G$
 $G \subset G$

(iv) G Abelian $\Rightarrow H \subset G$ also Abelian.

Examples

(i) $\mathbb{Z}_n \subset D_n$, $\mathbb{Z}_n = \{R_0, R_1, \dots, R_{n-1}\}$

(ii) $\mathbb{Z}_n \subset \mathbb{Z}_{nm}$
 $\parallel$

$$\{[0], [m], [2m], \dots, [(n-1)m]\}.$$

(iii) $U(1) \subset U(n)$
 $\parallel$

$$\{e^{i\psi} \mathbb{1}_n \mid \psi \in \mathbb{R}\}$$

(iv) $SO(n) \subset O(n)$

$$SU(n) \subset U(n)$$

(v) $O(n-1) \subset O(n)$

$$\begin{pmatrix} A & \\ & 1 \end{pmatrix}$$

(vi) $\forall g \in G: H = \Gamma_g = \{g^i \mid i \in \mathbb{Z}\}$

Abelian

(vii) center $Z(G) := \{h \in G \mid hg = gh, \forall g \in G\}$

$$Z(O(n)) = \{\pm \mathbb{1}_n\} \cong \mathbb{Z}_2$$

$$Z(\text{Dera}) = \{R_0, R_{n/2}\} = \mathbb{Z}_2$$

Def: $H \subset G$: left-coset $g \cdot H = \{g \cdot h \mid h \in H\} \subset G$



Rem: (i) gH is the rest class of equiv. relation.

$$a \sim b : \Leftrightarrow a^{-1}b \in H.$$

[reflective, symmetric, transitive]

(ii) g_1H and g_2H are either disjoint or identical.

$$g \in g_1H \cap g_2H \Rightarrow g \cdot h_1 = g = g_2 \cdot h_2$$

$$g_1 = g_2 h_2 h_1^{-1} \in g_2H$$

(iii) $g \cdot H$ is a ~~subset~~ subgroup $\Leftrightarrow g \in H$

(iv) $(gH) \mapsto (gH)^{-1} = Hg^{-1}$ left-coset $\leftrightarrow$ right coset

Examples:

(i) $G = (\mathbb{Z}, +)$

$$H = m\mathbb{Z}$$

cosets $m\mathbb{Z}, m\mathbb{Z} + 1, m\mathbb{Z} + 2, \dots, m\mathbb{Z} + (m-1)$

(ii) $G = D_3, H \cong \mathbb{Z}_3$

$$R_0 \mathbb{Z}_3, S_1 \mathbb{Z}_3 = S_1 \mathbb{Z}_3 = S_2 \mathbb{Z}_3$$

$$\{R_0, R_1, R_2\}$$

Def: $g_1, g_2 \in G$ are conjugate $g_1 \sim g_2$:

$$\exists g \in G : g_1 = g g_2 g^{-1}$$

This is an equiv. relation:

$$\text{left cosets } C_g = [g] = \{ \tilde{g} g \tilde{g}^{-1} \mid \tilde{g} \in G \} \quad \text{Conjugacy classes}$$

Example: $D_n : C_{R_i} = \{ R_{i+2j} \mid j \in \mathbb{Z} \}$

$$C_{S_i} = \{ S_{i+2j} \mid j \in \mathbb{Z} \}$$

Def: A subgroup $H \leq G$ is normal $H \triangleleft G$, if it is self conjugate, i.e.,

$$g H g^{-1} = H \quad \forall g \in G$$

Remark: (i) $H \triangleleft G, gH = Hg$ (left = right cosets)

(ii) H is union of conjugacy classes

(iii) $N \subset H \subset G, N \triangleleft G \Rightarrow N \triangleleft H$

but in general $N \triangleleft H \triangleleft G \not\Rightarrow N \triangleleft G$ WARNING!

Examples:

(i) center $Z(G)$

(ii) $\varphi: G \rightarrow G'$ hom.

$$H := \ker(\varphi) = \{ g \in G \mid \varphi(g) = e_{G'} \} \triangleleft G$$

Subgroup: $\varphi(g) = e = \varphi(h)$

$$\forall g, h \in H \Rightarrow \varphi(g) \cdot \varphi(h^{-1}) = \varphi(gh^{-1}) \Rightarrow gh^{-1} = H$$

$$\text{normal: } \varphi(ghg^{-1}) = \varphi(g) \underbrace{\varphi(h)}_{=e} \varphi(g^{-1}) = \varphi(g) \varphi(g^{-1}) = \varphi(gg^{-1}) = e$$

$$\Rightarrow ghg^{-1} \in H \quad \forall g \in G$$

e.g. $\det: U(n) \rightarrow U(1)$ hom.

$$\text{Ker}(\det) = \{A \in U(n) \mid \det(A) = 1\} = SU(n) \Rightarrow SU(n) \triangleleft U(n)$$

(iii) image of normal subgroups are normal in the image of the group.

$$H \triangleleft G \rightarrow \psi(H) \triangleleft \psi(G) \subset G'$$

$$\psi: G \rightarrow G'$$

Quotient group:

If $H \triangleleft G \rightarrow G/H = \{gH \mid g \in G\}$ forms a group

$$gH g'H = gg' \underbrace{H g'^{-1} g'}_e H = gg' H \cdot H = gg' H$$

well defined multip. map:

(i) associative $\mathbb{H}$

(ii) identity: H

(iii) inverse $(gH)^{-1} = g^{-1}H$

Def: Simple groups G don't have normal subgroups (building blocks)

Fact: $\psi: G \rightarrow G'$

$$H: \text{Ker}(\psi) \triangleleft G$$

$$G/H = G/\text{Ker}(\psi) \rightarrow G'$$

$$g \text{Ker}(\psi) \mapsto \psi(g)$$

$$f: G/H \xrightarrow{\cong} \psi(G) \text{ isomorphism}$$

e.g. $\psi = \det: U(n) \rightarrow U(1)$, $\text{Ker}(\psi) = SU(n)$

$$U(n)/SU(n) \cong U(1)$$

Product of groups:

Direct product: $G_1 \times G_2$

$$(g_1, g_2), (g'_1, g'_2) \in G_1 \times G_2$$

$$(g_1, g_2) \cdot (g'_1, g'_2) = (g_1 g'_1, g_2 g'_2) \rightarrow \text{satisfy group axioms.}$$

$$\pi_2: G_1 \times G_2 \rightarrow G_2 \quad (\text{this is a group homomorphism})$$

$$(g_1, g_2) \mapsto g_2$$

Also true for G_1 :

$$\text{Ker}(\pi_2) = G_1 \triangleleft G_1 \times G_2$$

$$G_2 \triangleleft G_1 \times G_2$$

Fact : $G = N_1 \times N_2 \iff$

- (i) $N_1 \triangleleft G$
- (ii) $N_1 \cap N_2 = \{e\}$
- (iii) $G = N_1 N_2$

Def. : semi-direct product :

N, H groups

$$\varphi : H \rightarrow \text{Aut}(N) = \text{Iso}(N)$$

$$N \rtimes_{\varphi} H = N \times H$$

as set

$$(n, h), n \in N, h \in H$$

$$(n, h) \cdot (n', h') = (n \cdot \varphi(h)(n'), hh')$$

if $\varphi = \text{identity}$, then this is the direct product.

$$e = (e_N, e_H)$$

$$(n, h)^{-1} = (\varphi(h)^{-1}(n^{-1}), h^{-1})$$

Check associativity:

$$((n, h)(n', h'))(n'', h'') = (n \varphi(h)(n'), hh')(n'', h'') =$$

$$= (n \varphi(h)(n') \varphi(hh')(n''), hh'h'')$$

$$\varphi(h)(\varphi(h')(n''))$$

φ is homom.

$$= (n \varphi(h)(n' \varphi(h')(n'')), hh'h'') =$$

$$= (n, h)((n', h')(n'', h''))$$

Examples: (i) $E(n) = \mathbb{R}^n \rtimes_{\varphi} O(n)$

$$\varphi : O(n) \rightarrow \text{Iso}(\mathbb{R}^n)$$

$$A \mapsto \{a \in \mathbb{R}^n \mapsto A \cdot a\}$$

$$(ii) G = D_n \quad N = \mathbb{Z}_n \triangleleft D_n$$

$$\{R_0, S_0\} = H = \mathbb{Z}_2$$

$$\varphi : \mathbb{Z}_n \rtimes \mathbb{Z}_2 \rightarrow D_n$$

$$(R_i, R_0 = e) \mapsto R_i R_0 = R_i$$

$$(R_i, S_0) \mapsto R_i S_0 = S_i$$

$$\begin{aligned}
 \varphi(R_i, a) \varphi(R_j, b) &= R_i a R_j b \\
 &= R_i \underbrace{a R_j a^{-1}}_{R_j'} a b = \varphi(R_i \varphi(a)(R_j), ab) \\
 R_j' &= a R_j a^{-1} \\
 &= \varphi(a)(R_j)
 \end{aligned}$$

$$\begin{aligned}
 \varphi: \mathbb{Z}_2 &\rightarrow \text{Iso}(\mathbb{Z}_n) \\
 a &\mapsto (R_i \mapsto a R_i a^{-1})
 \end{aligned}$$

22/04/15

Next lecture: 13.05.15

Problem Set!!!

Summary of last lecture:

Define basic concepts of groups

- groups
- subgroups
- homomorphisms
- conjugation
- normal groups, quotient groups
- cosets
- (semi-direct) products