

3. (LINEAR) REPRESENTATION

13/5/15

Def: A (linear) representation of a group G is a group homomorphism:

$$\rho: G \rightarrow \boxed{GL(V)} = \{ \text{invertible, linear transf. of } V \}$$

$\uparrow$
 repr. space

$\xrightarrow{\text{general linear group on } V}$
 group of all automorphisms of V , i.e. set of all bijective linear transformations $V \rightarrow V$ together with functional composition as group operation.

isomorphism from an object to itself ~ "symmetry group of the object"
 loosely speak.

V : vector space

Remark: $\dim(V) = n < \infty$ $\leftarrow$ dimension of the representation

then by choosing a basis, one can think of a representation as a collection of invertible $n \times n$ -matrices $\rho(g)$, $g \in G$

$$\text{s.t. } \rho(e) = \mathbb{1}_n$$

$$\rho(g)\rho(g') = \rho(gg') \quad \forall g, g' \in G$$

Examples

(i) $G = \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ representations on $V = \mathbb{C}$

$$GL(V) = \mathbb{C}^+ = \mathbb{C} \setminus [0] \rightarrow \dim(V) = n = 1$$

$\leftarrow$ complex without zero

$$\rho([a]) = e^{\frac{2\pi i a}{n}}$$

$$\rho([a])\rho([b]) = e^{\frac{2\pi i}{n}(a+b)} = \rho([a+b])$$

(ii) $V = \mathbb{R}^2$

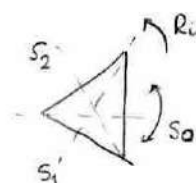
$$\rho([a]) = \begin{pmatrix} \cos(\frac{2\pi}{n}a) & \sin(\frac{2\pi}{n}a) \\ -\sin(\frac{2\pi}{n}a) & \cos(\frac{2\pi}{n}a) \end{pmatrix}$$

(iii) $G = D_n$, $V = \mathbb{R}^2$ $\dim(V) = n = 2$

$$\rho(R_j) = \begin{pmatrix} \cos(\frac{2\pi j}{n}) & \sin(\frac{2\pi j}{n}) \\ -\sin(\frac{2\pi j}{n}) & \cos(\frac{2\pi j}{n}) \end{pmatrix}$$

$$\rho(S_0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\rho(S_j) = \begin{pmatrix} \cos(\frac{2\pi j}{n}) & -\sin(\frac{2\pi j}{n}) \\ -\sin(\frac{2\pi j}{n}) & -\cos(\frac{2\pi j}{n}) \end{pmatrix}$$



$$\begin{aligned} R_i R_j &= R_{i+j} \\ R_i S_j &= S_{i+j} \\ S_i R_j &= S_{i-j} \\ S_i S_j &= R_{i-j} \end{aligned}$$

$$\rho(R_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$$

$$\rho(R_{a+b})$$

$$\rho(R_a)\rho(R_b) = \begin{pmatrix} \cos a & \sin a \\ -\sin a & \cos a \end{pmatrix} \begin{pmatrix} \cos b & \sin b \\ -\sin b & \cos b \end{pmatrix} = \begin{pmatrix} \cos a \cos b - \sin a \sin b & \cos a \sin b + \sin a \cos b \\ -\sin a \cos b - \cos a \sin b & -\sin a \sin b + \cos a \cos b \end{pmatrix} = \begin{pmatrix} \cos(a+b) & \sin(a+b) \\ -\sin(a+b) & \cos(a+b) \end{pmatrix} = \rho(R_{a+b})$$

$$\rho(R_a)\rho(S_b) = \begin{pmatrix} \cos a & \sin a \\ -\sin a & \cos a \end{pmatrix} \begin{pmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{pmatrix} = \begin{pmatrix} \cos a \cos b - \sin a \sin b & -\cos a \sin b + \sin a \cos b \\ -\sin a \cos b - \cos a \sin b & \sin a \sin b + \cos a \cos b \end{pmatrix} = \begin{pmatrix} \cos(a+b) & -\sin(a+b) \\ -\sin(a+b) & -\cos(a+b) \end{pmatrix} = \rho(S_{a+b})$$

◦ Every group has a trivial representation:

$$\dim(V) = 1 \quad \rho(g) = 1 \quad \leftarrow \text{maps everything to one}$$

$$\rho(g)\rho(g') = \rho(gg')$$

^{this}
 $\leadsto$ a representation does not contain any information about the group.

◦ Representation containing all information about G are called faithful:

$$\rho: G \hookrightarrow GL(V), \text{ which is injective}$$

no information has been lost

$$\text{This means: } \rho(g) = \rho(g') \Rightarrow g = g'$$

if it is not true: $\exists g \neq e \quad \rho(g) = \text{id}_V$ (identity in V) $\leftarrow$ degenerate representation

$$\text{Recall: } \ker(\rho) \triangleleft G$$

$\leadsto \rho$ descends to a faithful representation of $\boxed{G/\ker(\rho)}$

$$\text{e.g. } G = S_n$$

(i) has a trivial one-dimensional representation, $\rho(g) = 1$

(ii) has a non-trivial one-dim repr.: sign-representation

$$\rho(g) = (-1)^{\text{length}(g)} \quad [\text{remember: } (-1)^{\text{length}(g)}: S_n \rightarrow \mathbb{Z}_n]$$

$$\ker(\rho) = A_n \text{ alternating group}$$

ρ descends to a faithful representation of $S_n/A_n = \mathbb{Z}_n$

◦ Building representations:

$$\begin{aligned} \text{Given a representation } \rho_1: G &\longrightarrow GL(V_1) \\ \rho_2: G &\longrightarrow GL(V_2) \end{aligned}$$

* All the representations that we will consider are representations of a GL of a vector space.

(i) Direct sum:

$$\rho_1 \oplus \rho_2(g) = \rho_1(g) \oplus \rho_2(g) = \left(\begin{array}{c|c} \rho_1(g) & 0 \\ \hline 0 & \rho_2(g) \end{array} \right)$$

$$V_{\rho_1 \oplus \rho_2} = V_1 \oplus V_2$$

$$\dim(\rho_1 \oplus \rho_2) = \dim(\rho_1) + \dim(\rho_2)$$

$\uparrow$
 dim. of the representation - vector space

(ii) Tensor product: $V_{\beta_1} \otimes V_{\beta_2} = V_{\beta_1} \otimes V_{\beta_2}$

$$\rho_1 \otimes \rho_2(g) = \rho_1(g) \otimes \rho_2(g)$$

$$\dim(\rho_1 \otimes \rho_2) = \dim(\rho_1) \cdot \dim(\rho_2)$$

(iii) Dual representation

$$j: G \rightarrow GL(V)$$

$$\text{dual vs. } V^* := \text{Hom}(V, \mathbb{K})$$

$$\bar{j}: G \rightarrow GL(V^*)$$

$$\bar{j}(g) = (j(g^{-1}))^*$$

↑ dual map

$\bar{j}$ is representation

$$\bar{j}(g) \bar{j}(h) = (j(g^{-1}))^* (j(h^{-1}))^* =$$

$$= (j(h^{-1}) j(g^{-1}))^* =$$

$$= (j(h^{-1} g^{-1}))^* =$$

$$= (j((gh)^{-1}))^* = \bar{j}(gh)$$

About dual maps

$$A: V \rightarrow W$$

$$A^*: V^* \rightarrow W^*$$

$$W^* = \text{Hom}(W, \mathbb{K}) \ni \omega^*$$

$$A^*(\omega^*) = \omega^* \circ A: V \rightarrow \mathbb{K}$$

$$\in \text{Hom}(V, \mathbb{K}) = V^*$$

Prop of dual maps:

$$A: V \rightarrow W$$

$$B: W \rightarrow U$$

$$(BA)^* = A^* B^*$$

- Subrepresentations:

Def: ^{An} invariant subspace W of a representation $j: G \rightarrow GL(V)$ is a subspace $W \subset V$, s.t.

$$j(g)(W) \subset W$$

$\Rightarrow j(g)|_W$ is a subrepresentation.

$j|_W: G \rightarrow GL(W)$ is a representation on its own.

Def: j is called irreducible if there are no invariant subspaces, except $\{0\}$ and V .

Otherwise (if it's not irreducible) it is called reducible.

In matrix form this means:

$$j(g) = \left(\begin{array}{c|c} j_{W,W}(g) & * \\ \hline 0 & * \end{array} \right) \quad \swarrow \text{reducible representation}$$

(in a particular basis)

• A repr. ρ is called decomposable if it is a direct sum $\rho = \rho_1 \oplus \rho_2$

In matrix form :
(in a particular basis)

$$\left(\begin{array}{c|c} \rho_1(g) & 0 \\ \hline 0 & \rho_2(g) \end{array} \right) = \rho(g)$$

If it is not decomposable, it is called indecomposable.

• A repr. is called fully decomposable if it can be decomposed in a direct sum of irreducible repr. :

$$\rho = \rho_1 \oplus \dots \oplus \rho_n$$

← irreducible representation

Ex: $\rho: \mathbb{Z} \rightarrow GL(2)$

$$\rho(n) \mapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \quad \rho(n)\rho(m) = \rho(n+m)$$

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & m+n \\ 0 & 1 \end{pmatrix}$$

There is an invariant subspace reducible

$$\mathbb{R}^2 = \text{span}(e_1, e_2), \quad W = \text{span}(e_1) \text{ is invariant}$$

But it is not decomposable: because $\rho(n)$ cannot be diagonalised

$\rho|_W$ is the trivial representation

(ii) $G = S_3$, $V = \text{span}\{e_1, e_2, e_3\}$

$$\rho(\pi)(e_i) = e_{\pi(i)}$$

$$\text{In basis: } \rho((12)) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\rho((23)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Invariant subspace: $W = \text{span}\{e_1 + e_2 + e_3\}$, because

$$\rho(\pi)(e_1 + e_2 + e_3) = e_{\pi(1)} + e_{\pi(2)} + e_{\pi(3)} = e_1 + e_2 + e_3$$

$$W^\perp = \text{span}\{V_1 = e_1 - e_2, V_2 = e_2 - e_3\}$$

$$\rho|_{W^\perp}((12)): \begin{array}{l} V_1 \mapsto -V_1 \\ V_2 \mapsto V_1 + V_2 \end{array} \quad \text{In matrix form} \quad \rho((12)) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\rho|_{W^\perp} : \begin{array}{l} V_1 \mapsto V_1 + V_2 \\ V_2 \mapsto -V_2 \end{array} \quad \text{In matrix form: } \rho((23)) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

$\rightarrow \rho|_{W^\perp}$ is irreducible

$$\rho = \rho|_W \oplus \rho|_{W^\perp} \quad \sim \text{Here we had a repr.} \rightarrow \text{irreducible}$$

$\uparrow$ trivial $\uparrow$ 2dim irred. repr. $\rightarrow$ fully decomposable

Examples: for building of representations

$$\rho_r : \mathbb{Z}_n \rightarrow GL(\mathbb{C})$$

$$\rho_r([a]) = e^{\frac{2\pi i r a}{n}}$$

$$(i) \rho_r \oplus \rho_s([a]) = \begin{pmatrix} e^{\frac{2\pi i r a}{n}} & 0 \\ 0 & e^{\frac{2\pi i s a}{n}} \end{pmatrix}$$

direct sum

(ii) tensor product

$$\mathbb{C} \otimes \mathbb{C} = \mathbb{C} \leftarrow \text{we'll get again a 1-dim representation}$$

$$\rho_r \otimes \rho_s([a]) = \rho_r([a]) \otimes \rho_s([a]) = e^{\frac{2\pi i (r+s)a}{n}} = \rho_{r+s}([a])$$

(iii) dual representation

$$\mathbb{C}^* = \mathbb{C}$$

$$\bar{\rho}_r([a]) = \rho_r([a]^{-1}) = \rho_r([-a]) = e^{-\frac{2\pi i r a}{n}} = \rho_{-r}([a])$$

$$\bar{\rho}_r = \rho_{-r}$$

$$\rho_r \otimes \rho_s = \rho_{r+s}$$

Special repr. compatible with some structures:

- Unitary representation:

Def: A unitary representation ρ is a repr. on a Hermitian vector space $(V, \langle, \rangle)$

s.t. $g: G \rightarrow \mathcal{U}(V) = \{T \in \mathcal{U}(V), (\langle a, b \rangle = \langle b, a \rangle)\}$

$$\langle T_a, T_b \rangle = \langle a, b \rangle \quad \forall a, b \in V \quad \} \\ \subset GL(V)$$

$$(T^*)^t_F = T + T = \text{id}_V$$

Unitary representations are nice:

Fact: Unitary repr. are fully decomposable.

Assume $\rho: G \rightarrow U(V)$, $W \subset V$ is invariant subspace

$\Rightarrow W^\perp$ is invariant subspace as well.

$$W^\perp = \{v \in V \mid \langle v, w \rangle = 0 \quad \forall w \in W\}$$

$$\forall v \in W^\perp \quad : \quad 0 = \langle v, w \rangle = \langle v, \rho(g)w \rangle = \langle \rho(g^{-1})v, w \rangle$$

$\Rightarrow W^\perp$ is invariant ~~under~~ subspace as well

$$\Rightarrow g = g|_W \oplus |_{W^\perp}$$

Iteratively $\leadsto \rho = \bigoplus_i \rho_i$
 $\uparrow$ irreducible representation

→ building blocks of unitarity repres. are the irreducible repr.

◦ Maps between representations:

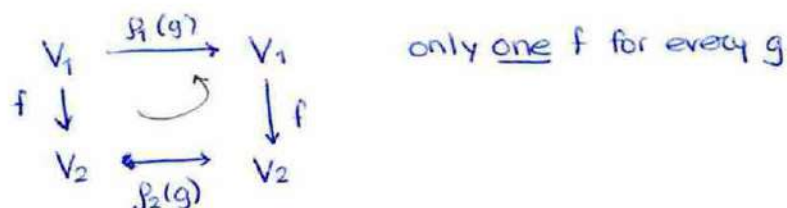
Def: An intertwiner between two representations:

$$\rho_1: G \rightarrow GL(V_1)$$

$$\rho_2: G \rightarrow GL(V_2)$$

is a linear map $f: V_1 \rightarrow V_2$

$$\text{s.t. } \rho_2(g) \circ f = f \circ \rho_1(g) \quad \forall g \in G$$



If f is invertible $\Rightarrow$ equivalence of representations

$$\rho_1(g) = f^{-1} \rho_2(g) f$$

↖ change of basis transf.

(requires $\dim(V_1) = \dim(V_2)$)

Remark: $\text{Im}(f)$ and $\text{Ker}(f)$ are invariant subspaces.

$$\begin{array}{c} \overline{\rho_2(g) f(v)} = f \circ \underbrace{\rho_1(g)(v)}_{\substack{\text{def of} \\ \text{intertwiner}}} \\ \in \text{Im}(f) \quad \uparrow \quad \in \text{Im}(f) \end{array} \Rightarrow \text{Im}(f) \text{ is invariant subspace.}$$

$$\circ f(v) = 0, \text{ i.e. } v \in \text{Ker}(f)$$

$$\begin{array}{c} \rho_2(g) f(v) \\ \parallel \\ 0 \end{array} = f \circ \rho_1(g)v \Rightarrow \rho_1(g)v \in \text{Ker}(f)$$

$\rightarrow \text{Ker}(f)$ is invariant subspace. $\perp$

- Schur's Lemma:

If ρ_1 and ρ_2 are irreducible repr. of a group G and $f: V_1 \rightarrow V_2$ is an intertwiner, then:

(i) either $f \equiv 0$ or f is an isomorphism.

(ii) if $V_1 \cong V_2 \Rightarrow f = \lambda \text{id}$, $\lambda \in \mathbb{C}$, (over $\mathbb{C}$)

Γ (i) $f: V_1 \rightarrow V_2$ is intertwiner

$\ker(f)$ is invariant subspace $\xrightarrow{\text{irreducibility of } \rho_2} \ker(f) = V_1 \text{ or } \ker(f) = \{0\}$
 $f \equiv 0$ f injective

$\text{Im}(f)$ is invariant subspace $\Rightarrow \text{Im}(f) = \{0\}$ or $\text{Im}(f) = V_2$
 $f \equiv 0$ surjective

This shows (i).

(ii) Over $\mathbb{C}$: since $\mathbb{C}$ is algebraically closed, f has at least one e-value λ .

We have already set 1

$V = V_1 = V_2$

$f' = (f - \lambda \text{id}_V)$ is still an intertwiner.

but now: $\ker(f') \neq \{0\}$ by irreducibility of $\ker(f') = V$

$\Rightarrow f' = (f - \lambda \text{id}) \equiv 0$

this shows (ii).

┘

Remark: If two irreducible representations ρ_1, ρ_2 over $\mathbb{C}$ are equivalent as in (ii), then the intertwiners are multiples of each other.

all

$f_1, f_2: V_1 \rightarrow V_2, f_2 \neq 0$

$\Rightarrow f_2^{-1} f_1: V_1 \rightarrow V_1$

$\lambda \text{id} \Rightarrow \boxed{f_1 = \lambda f_2}, \lambda \in \mathbb{C}$

Note $\hookrightarrow$ this works over $\mathbb{C}$!

Aside: $\mathbb{R}$ is not algebraically closed

$\leadsto$ not every intertwiner has eigenvalue.

$\leadsto$ arguments above does not ~~work~~ work.

$\leadsto$ Not all intertwiners over $\mathbb{R}$ are multiples of each other.

↳ This is the bad news. However $\mathbb{R}$ still have enough structure:

The intertwiners form an associative division algebra.

- can add them
- can multiply them
- if they are not zero, one can divide by them.
- since repr. by matrices, associative

◦ Division algebra over $\mathbb{C}$; $\mathbb{C}$

Associative

◦ Division algebras over $\mathbb{R}$: $\begin{cases} \mathbb{R} \\ \mathbb{C} \\ \mathbb{H}, \text{quaternions} \end{cases}$