

4. REPRESENTATIONS of FINITE GROUPS

$|G| < \infty$, complex representations.

Fact: Any representation of G is fully decomposable.

Let $W \subset V \leftarrow \text{repr.}$ is invariant subspace.

Choose a projector $\pi_0: V \rightarrow V$ onto W , i.e. $\text{Im}(\pi_0) = W$

Define:

$$\pi := \frac{1}{|G|} \sum_{g \in G} \rho(g) \pi_0 \rho(g^{-1}) : V \rightarrow V$$

i) π is still a projector on W

ii) π is an intertwiner

From this it follows $V = \underbrace{\text{Im}(\pi)}_W \oplus \text{Ker}(\pi)$
 $\swarrow$ π is a projector
 $\nwarrow$ is invariant subspace, because π is intertwiner.

$$i) \pi^2 = \frac{1}{|G|^2} \sum_{g, k \in G} \rho(g) \pi_0 \rho(g^{-1}) \rho(k) \pi_0 \rho(k^{-1}) =$$

$$= \frac{1}{|G|} \sum_{k \in G} \rho(k) \pi_0 \rho(k^{-1}) = \pi$$

$$\pi|_W = \text{id}_W$$

$$ii) \pi \circ \rho(h) = \frac{1}{|G|} \sum_{g \in G} \rho(g) \pi_0 \overbrace{\rho(g^{-1}) \rho(h)}^{\rho(g^{-1}h)} = (g \mapsto hg)$$

$$= \frac{1}{|G|} \sum_{g \in G} \overbrace{\rho(hg)}^{\rho(h)\rho(g)} \pi_0 \rho(g) =$$

$$= \rho(h) \pi$$

Any representation ρ of a finite group decomposes into a sum of irreducible representation:

$$\rho \cong \rho_1^{\oplus n_1} \oplus \dots \oplus \rho_k^{\oplus n_k}, \quad n_i \in \mathbb{N}_0$$

Remark: Proof by averaging over group $\rightarrow$ can be repeated for non-finite but compact groups, where instead of summing, one can integrate

Decomposition

start with representation $\rho: G \rightarrow GL(V)$

$$\pi := \frac{1}{|G|} \sum_{g \in G} \rho(g) \quad V \rightarrow V$$

• it is a projector on the invariant subspace.

$$V^G := \{v \in V \mid \rho(g)v = v \quad \forall g \in G\}$$

$$\begin{aligned} (i) \quad \pi^2 &= \frac{1}{|G|^2} \sum_{g, k \in G} \rho(g) \rho(k) \quad (g \mapsto gh^{-1}) \\ &\quad \underbrace{\rho(gk)}_{\rho(gh)} \\ &= \frac{1}{|G|} \sum_{g \in G} \rho(g) = \pi \end{aligned}$$

$$(ii) \quad v \in V^G: \pi v = v \Rightarrow \pi|_{V^G} = \text{id}_{V^G} \Rightarrow V^G \subset \text{Im}(\pi)$$

$$\rho(g)\pi w = \rho(g) \frac{1}{|G|} \sum_{h \in G} \rho(h) w = \pi w$$

$$\Rightarrow \text{Im}(\pi) \subset V^G \Rightarrow V^G = \text{Im}(\pi)$$

$\rightarrow \pi$ projects on $V^G = \text{Im}(\pi)$

We want to know the dim of V^G .

$$\dim(V^G) = \text{Tr}_V(\pi) = \frac{1}{|G|} \sum_{g \in G} \underbrace{\text{Tr}_V(\rho(g))}_{\chi_\rho(g)}$$

$\chi_\rho(g) = \chi_V(g)$ Character of the representation ρ, V

$$= \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g)$$

Apply this

$$\text{Hom}(V_1, V_2) \cong V_1^* \otimes V_2 \quad \text{carries reprs. } \bar{\rho}_1 \otimes \rho_2$$

$$\bar{\rho}_1 \otimes \rho_2(g): f \mapsto \rho_2(g) \circ f \circ \rho_1(g^{-1})$$

$$\begin{aligned} \text{Hom}(V_1, V_2)^G &= \{f \in \text{Hom}(V_1, V_2) \mid \rho_2(g) \circ f \circ \rho_1(g^{-1}) = f\} = \\ &= \text{space of intertwiners} \end{aligned}$$

If ρ_1 and ρ_2 are irreducible, by Schur's lemma

$$\dim(\text{Hom}(V_1, V_2)^G) = \begin{cases} 1, & \rho_1 \cong \rho_2 \\ 0 & \text{otherwise} \end{cases}$$

Then:

$$\dim \frac{1}{|G|} \sum_{g \in G} \chi_{\bar{\rho}_1 \otimes \rho_2}(g) = \begin{cases} 1, & \rho_1 \cong \rho_2 \\ 0 & \text{otherwise} \end{cases}$$

"

$$\frac{1}{|G|} \sum_{g \in G} \text{Tr}_{V_1^* \otimes V_2}(\bar{\rho}_1(g) \otimes \rho_2(g))$$

" $\bar{\rho}_1(g) = \rho_1(g^{-1})$ on V_1

$$\frac{1}{|G|} \sum_{g \in G} \text{Tr}_{V_1^*}(\bar{\rho}_1(g)) \cdot \underbrace{\text{Tr}_{V_2}(\rho_2(g))}_{\chi_2(g)}$$

"

$$\frac{1}{|G|} \sum_{g \in G} \chi_1(g^{-1}) \chi_2(g)$$

$$\frac{1}{|G|} \sum_{g \in G} \chi_i(g^{-1}) \cdot \chi_j(g) = \delta_{ij}$$

sum over irreducible representations

$$V = V_1^{\oplus n_1} \oplus \dots \oplus V_k^{\oplus n_k}$$

$$\text{Hom}(V_i, V) = \oplus \text{Hom}(V_i, V_a)^{\oplus n_a}$$

$$n_i = \text{Tr}_{\text{Hom}(V_i, V)}(\pi_i) = \frac{1}{|G|} \sum_{g \in G} \chi_{\rho_i}(g^{-1}) \chi_{\rho}(g)$$

From Characters one can obtain the decomposition of V .

Can do more: identify the ~~representative repr~~ respective subrepresentation

Next step: how to localize a subgroup V_i in ~~among~~ V

Projector on $V_1^{\oplus n_1}$

$$\pi_{\rho_1} := \frac{\dim(V_1)}{|G|} \sum_{g \in G} \chi_{\rho_1}(g^{-1}) \rho(g) : V \rightarrow V$$

i) π_{ρ_1} is intertwiner:

$$\begin{aligned} \pi_{\rho_1} \circ \rho(h) &= \frac{d_1}{|G|} \sum_{g \in G} \chi_{\rho_1}(g^{-1}) \rho(g h) = \\ &= \frac{d_1}{|G|} \sum_{g \in G} \chi_{\rho_1}(h g^{-1} h^{-1}) \rho(h g) = \\ &\quad \chi_{\rho_1}(g^{-1}) \xleftarrow{\text{cyclicality of trace}} \chi_{\rho_1}(h g^{-1} h^{-1}) = \text{Tr}(\rho(h g^{-1} h^{-1})) = \\ &= \text{Tr}(\rho(h) \rho(g^{-1}) \rho(h^{-1})) = \\ &= \text{Tr}(\rho(h^{-1}) \rho(h) \rho(g^{-1})) = \text{Tr}(\rho(g^{-1})) \\ &= \rho(h) \pi_{\rho_1} \end{aligned}$$

ii) π_{ρ} is a projector:

$$\begin{aligned} \pi_{\rho}^2 &= \frac{d_1^2}{|G|^2} \sum_{g, h \in G} \chi_{\rho_1}(g^{-1}) \chi_{\rho_1}(h^{-1}) \rho(g h) \quad g \mapsto g h^{-1} \\ &= \frac{d_1^2}{|G|^2} \sum_{g, h} \chi_{\rho_1}(h g^{-1}) \chi_{\rho_1}(h^{-1}) \rho(g) = \quad \left[\text{Use Ex. 3c) from PS 2} \right] \\ &= \frac{d_1}{|G|} \sum_{g \in G} \chi_{\rho_1}(g^{-1}) \rho(g) = \pi_{\rho_1} \quad \sum_h \chi_{\rho_1}(h g^{-1}) \chi_{\rho_1}(h^{-1}) = \chi_{\rho_1}(g^{-1}) \frac{|G|}{d_1} \delta_{1, g} \end{aligned}$$

iii) π_{ρ_1} projects onto $V_1^{\oplus n_1} \subset V$

Look at representation $\text{Im}(\pi_{\rho_1}) = V_1^{\oplus n_1} \oplus \dots \oplus V_k^{\oplus n_k}$

Want to show that $n_1 = n_1$

$$\begin{aligned} n_1 &= \frac{1}{|G|} \sum_{g \in G} \chi_{\rho_1}(g^{-1}) \chi_{\text{Im}(\pi_{\rho_1})}(g) = \\ &\quad \text{Tr}_{\text{Im}(\pi_{\rho_1})} \rho(g) = \text{Tr}_V(\pi_{\rho_1} \rho(g)) \\ &= \frac{d_1}{|G|} \sum_{g \in G} \chi_{\rho_1}(h^{-1}) \underbrace{\text{Tr}_V(\rho(h) \rho(g))}_{\chi_V(hg)} = \frac{d_1}{|G|} \sum_{h \in G} \chi_{\rho_1}(h^{-1}) \chi_V(hg) \end{aligned}$$

$$n_i' = \frac{d_1}{|G|^2} \sum_{g,h} \chi_{\rho_1}(g^{-1}) \chi_{\rho_2}(h^{-1}) \chi_{\nu}(hg) \quad g \mapsto h^{-1}g$$

$$= \frac{d_1}{|G|^2} \sum_{g,h} \chi_{\rho_1}(g^{-1}h) \chi_{\rho_2}(h^{-1}) \chi_{\nu}(g) =$$

$$= \delta_{i,1} \underbrace{\frac{1}{|G|} \sum_{g \in G} \chi_{\rho_1}(g^{-1}) \chi_{\nu}(g)}_{n_i} =$$

Ex 3.

$$= \delta_{i,1} n_i$$

$$\rightarrow \pi_{\rho_1} \text{ projects onto } V_1^{\oplus n_i} \subset V$$

- characters of irreducible representations.

$$\chi_{\rho}(g) = \text{Tr}(\rho(g))$$

Properties i) $\chi_{\rho_1 \oplus \rho_2}(g) = \chi_{\rho_1}(g) + \chi_{\rho_2}(g)$

ii) $\chi_{\rho_1 \otimes \rho_2}(g) = \chi_{\rho_1}(g) \cdot \chi_{\rho_2}(g)$

iii) $\chi_{\rho}(e) = \dim(\rho)$ ← dimension of the representation space

iv) $\chi_{\rho}(g^{-1}) = \overline{\chi_{\rho}(g)}$

v) $\chi_{\rho}(h^{-1}gh) = \chi_{\rho}(g)$ ✓ by cyclicity of trace
 χ_{ρ} only depends on conjugacy class of g !

→ χ_{ρ} is a class function.

Space of class functions:

$$\mathbb{C}^{\text{class}}(G) = \{ f: G \rightarrow \mathbb{C} \mid f(h^{-1}gh) = f(g) \}$$

is a vector space.

$$\dim(\mathbb{C}^{\text{class}}(G)) = \# \text{ conjugacy classes in } G$$

On this space define a Hermitian form

$$(\alpha, \beta) = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \beta(g)$$

$$\chi_i := \chi_{\rho_i} \quad \rho_i \text{ irreducible} \quad \in \mathbb{C}^{\text{class}}(G)$$

$$(\chi_i, \chi_j) = \delta_{i,j} \quad \text{orthonormal set}$$

$$\rightarrow \# \text{ irred repres} \leq \# \text{ conjugacy classes} \quad 16$$

Want to show: χ_i an ON basis of $\mathbb{C}^{\text{class}}(G)$

χ_i are linearly independent

→ characterize representation completely.

Γ We have already seen that χ_i are an ON set in $\mathbb{C}^{\text{class}}(G)$

If it is a basis: if $(\alpha, \chi_j) = 0 \quad \forall j \Rightarrow \alpha = 0$

Assume $(\alpha, \chi_i) = 0 \quad \forall i$

$$\psi_{\alpha}^{j_i} := \sum_g \overline{\alpha(g)} f_i(g) \cdot V_i \rightarrow V_i.$$

This is an intertwiner:

$$\begin{aligned} \psi_{\alpha}^{j_i} f_i(h) &= \sum_g \overline{\alpha(g)} f_i(gh) \quad g \mapsto hgh^{-1} \\ &= \sum_g \overline{\alpha(hgh^{-1})} f_i(hg) \\ &= f_i(h) \sum_g \overline{\alpha(hgh^{-1})} f_i(g) = \\ &\quad \underbrace{\alpha(g)} \\ &= f_i(h) \psi_{\alpha}^{j_i} \rightarrow \text{intertwiner} \end{aligned}$$

V_i irreducible: by Schur's lemma: $\psi_{\alpha}^{j_i} = \lambda_i \text{id } V_i$

$$\lambda_i = \frac{\text{Tr}(\psi_{\alpha}^{j_i})}{\dim(V_i)} = \frac{1}{d_i} \sum_g \overline{\alpha(g)} \chi_{j_i}(g)$$

$$= \frac{|G|}{d_i} (\alpha, \chi_{j_i}) = 0 \Rightarrow \boxed{\psi_{\alpha}^{j_i} = 0} \quad \forall i$$

$$\Rightarrow \sum_g \overline{\alpha(g)} f_i(g) = 0 \quad \forall i$$

$$\Rightarrow \sum_g \overline{\alpha(g)} f(g) = 0 \quad \forall \text{ repres. } f$$

Find a representation f such that all $f(g_i)$ are linearly independent.

$$\Rightarrow \overline{\alpha(g)} = 0 \quad \forall g$$

A representation for which this holds is the so-called regular represent.

$$V = \mathbb{C}[G] = \left\{ \sum_i \alpha_i g_i \mid \alpha_i \in \mathbb{C} \right\}$$

group algebra of G .

$$\rho(g) \left(\sum \alpha_i g_i \right) = \sum \alpha_i (g g_i)$$

$$\sum \alpha_i \rho(g_i) g = \sum \alpha_i (g_i g) = 0 \text{ only if } \alpha_i = 0 \forall i$$

$\rightarrow \rho(g_i)$ are linearly independent.

$\rightarrow \chi_i$ is an orthonormal basis of $\mathbb{C}^{\text{class}}[G]$, $\dim \mathbb{C}^{\text{class}}[G] = \# \text{ conj. classes of } G$

$\rightarrow$ as many irreducible representations as there are conjugacy classes

$\rightarrow$ Characters determine repr.

* Characters table

		conjugacy class of the identity			
G		C_1	C_2	...	C_k
ρ_1	$\chi_1(G)$ d_1	$\chi_1(C_1)$	$\chi_1(C_2)$		
$\vdots$					
ρ_k	$\chi_k(G)$ d_k	$\chi_k(C_1)$	...		$\chi_k(C_k)$

$k = \# \text{ conj. - classes of } G$

$$\chi_{ia} := \chi_i(C_a)$$

$$\delta_{ij} = \frac{1}{|G|} \sum_g \overline{\chi_i(g)} \chi_j(g) = \sum_a \overline{\chi_i(C_a)} \chi_j(C_a) \frac{|C_a|}{|G|}$$

orthogonality of rows

$$D_{a,b} := \frac{|C_a|}{|G|} \delta_{a,b}$$

$$X D \bar{X}^T = \mathbb{1}_n \Rightarrow (\bar{X}^T) X = D^{-1}$$

$$\Rightarrow \sum_i \bar{\chi}_i(C_a) \cdot \chi_i(C_b) = \delta_{a,b} \frac{|G|}{|C_a|}$$

orthogonality of columns

Constrains character table

Another constraint from regular representation.

$$\rho^{\text{reg}}(g) (\sum \alpha_i g_i) = \sum \alpha_i (g g_i)$$

$$\chi_{\text{reg}}(g) = \text{Tr}_{\mathcal{C}[G]}(\rho^{\text{reg}}(g)) = \# \{ i \mid g \cdot g_i = g_i \} = \begin{cases} |G|, & g=e \\ 0 & \text{otherwise} \end{cases}$$

$\uparrow$
 basis g_i
 $\rho(g)g_i = (g \cdot g_i)$

$$\rho^{\text{reg}} = \rho_1^{\oplus n_1} \oplus \dots \oplus \rho_k^{\oplus n_k}$$

$$n_i = (\chi_{\rho_i}, \chi_{\text{reg}}) = \frac{1}{|G|} \sum_{g \in G} \chi_{\rho_i}(g^{-1}) \chi_{\text{reg}}(g) =$$

$$= \frac{1}{|G|} \dim(\rho_i) \cdot |G| = \dim(\rho_i)$$

$$\Rightarrow \rho^{\text{reg}} = \rho_1^{\oplus d_1} \oplus \dots \oplus \rho_k^{\oplus d_k}, \quad d_i = \dim(\rho_i)$$

$$\dim(\rho^{\text{reg}}) = \dim(\mathcal{C}[G]) = |G| = \sum_i d_i^2 \quad \Rightarrow \quad \boxed{|G| = \sum d_i^2}$$

$$6 = 1 + 1 + d_2^2 \Rightarrow d_2 = 2$$

Example $D_3 \cong S_3$

$|G|$

$$|D_3| = 6$$

Conjugacy classes

$$(\cdot) (\cdot) (\cdot) \quad \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \quad C_0 = C(R_0) = \{R_0, e\}$$

$$(\cdot \cdot) (\cdot) \quad \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \quad C_1 = C(S_0) = \{S_0, S_4, S_2\}$$

$$(\cdot \cdot \cdot) \quad \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} \quad C_2 = C(R_1) = \{R_1, R_3\}$$

3 irreducible representations

	C_0	C_1	C_2
D_3/S_3	1	3	2
ρ^{triv}	1	1	1
ρ^{sgn}	1	-1	1
$\rho_2 \oplus$	2	$\alpha=0$	$\beta=-1$
ρ^P	3	1	0

$$(*) \quad |G| = 2d_1^2$$

$$6 = 1 + 1 + d_2^2 \rightarrow d_2 = 2$$

Obtain α and β using orthogonality:

$$0 = (\chi^{\text{triv}}, \chi_2) = \frac{1}{6} (2 \cdot 1 + 3 \cdot \alpha + 2 \cdot \beta)$$

$$0 = (\chi^{\text{sgn}}, \chi_2) = \frac{1}{6} (2 - 3 \cdot \alpha + 2\beta)$$

$$\Rightarrow \alpha = 0, \beta = -1$$

Look at

$$\rho(S_0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho(R_1) = \begin{pmatrix} \cos(\frac{2\pi}{3}) & -\sin(\frac{2\pi}{3}) \\ \sin(\frac{2\pi}{3}) & \cos(\frac{2\pi}{3}) \end{pmatrix}$$

$$\chi_\rho(C_0) = 2 \quad \chi_\rho(C_1) = 0 \quad \chi_\rho(C_2) = 2\cos(\frac{2\pi}{3}) = -1$$

$\rho \cong \rho_2$ because the characters agree.

$\oplus \oplus$ Permutation representation.

$$\rho^P(e) = 1_{\mathbb{C}^3} \quad \chi_\rho(C_0) = 3$$

$$\rho^P(R_1) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \chi_\rho(C_1) = 1$$

$$\rho^P(S_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \chi_\rho(C_2) = 0$$

$$\rho^P = \rho^{\oplus n_1}_{\text{triv}} \oplus \rho^{\oplus n_2}_{\text{sgn}} \oplus \rho^{\oplus n_3}_2$$

$$n_1(\chi^{\text{triv}}, \chi_3) = \frac{1}{6} (3 \cdot 1 + 3 \cdot 1) = 1$$

$$n_2(\chi^{\text{sgn}}, \chi_3) = \frac{1}{6} (3 \cdot 1 - 3 \cdot 1) = 0$$

$$n_3(\chi_2, \chi_3) = \frac{1}{6} (3 \cdot 2) = 1$$

$$\boxed{\rho^P \cong \rho^{\text{triv}} \oplus \rho_2}$$

Projectors

$$\begin{aligned} \Pi_{\text{triv}} &= \frac{1}{6} \sum_g \overline{\chi^{\text{triv}}(g)} \rho^P(g) = \\ &= \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \rightarrow \text{projector on } \mathbb{C}(e_1 + e_2 + e_3) \end{aligned}$$

$$\begin{aligned} \Pi_2 &= \frac{2}{6} \left(2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \right) = \\ &= \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \end{aligned}$$

$\rightarrow$ projector on the space $\text{span}\{e_1 - e_2, e_2 - e_3\}$

27/05/15

Summary from last lecture:

- complex representation of finite groups are fully decomposable.

$$\rho = \bigoplus_i \rho_i \oplus n_i = \underbrace{\rho_1 \oplus \dots \oplus \rho_1}_{n_1} \oplus \underbrace{\rho_2 \oplus \dots \oplus \rho_2}_{n_2} \oplus \dots$$

irreducible
reps.

- as many irred. reps. as there are conjugacy classes in G

- reps. determined by characters: $\chi_\rho(g) = \text{Tr}_{V_\rho}(\rho(g))$

$(\chi_i = \chi_{\rho_i})$, irr. reps. are ON basis of class-fct

- $n_i = (\chi_i, \chi_\rho) = \frac{1}{|G|} \sum_{g \in G} \frac{\chi_i(g^{-1}) \chi_\rho(g)}{\chi_i(g)}$

- projector $\pi_i = \frac{d_i}{|G|} \sum_{g \in G} \chi_i(g^{-1}) \rho(g) : V \rightarrow V$ projects onto $V_i^{\oplus n_i} \subset V$

$$d_i \equiv \dim(\rho_i)$$

- Characters - table:

G	$C_1 = [e]$ $ C_1 = 1$	C_2 $ C_2 $	...	C_k $ C_k $
ρ_1	$d_1 = \chi_1(e)$ $\chi_1(g)$		...	χ_1
$\vdots$	$\vdots$			
ρ_k	$d_k = \chi_k(e)$		...	$\chi_k(g)$

$$\sum_a \frac{|C_a|}{|G|} \bar{\chi}_i(a) \chi_j(a) = \delta_{i,j}$$

$$\sum_i \frac{|C_a|}{|G|} \bar{\chi}_i(a) \chi_j(b) = \delta_{a,b}$$

$$\sum_i d_i^2 = |G|$$

$$n_0 = (\chi_0, \chi_3) = \frac{1}{6} (3 \cdot 1 \cdot 1 + 1 \cdot 1 \cdot 3) = 1$$

$$\frac{7}{32} \cdot 4 - 10 = 22$$

$$n_1 = (\chi_1, \chi_3) = \frac{1}{6} (3 - 3) = 0$$

$$15 - 7$$

$$n_2 = (\chi_2, \chi_3) = \frac{1}{6} \cdot 6 = 1$$

$$\boxed{\mathcal{P}_3 \cong \mathcal{P}_1 \oplus \mathcal{P}_2}$$

Projector:

$$\pi_0 = \frac{1}{6} \sum_{g \in G} \bar{\chi}_0(g) \mathcal{P}_3(g) = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \sim \text{Projection on span}\{e_1 + e_2 + e_3\}$$

$$V_0 \subset V_3 = \text{span}(e_1, e_2, e_3)$$

$$\in (e_1 + e_2 + e_3)$$

$$\pi_2 = \frac{1}{6} \sum_{g \in G} \bar{\chi}_2(g) \mathcal{P}_3(g) =$$

$$= \frac{1}{3} \left(2 \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \right) = \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \sim \text{Projection on } V_2 = \text{span}\{e_1 - e_2, e_2 - e_3\}$$

$$\mathcal{P}_4 = \mathcal{P}_2 \oplus \mathcal{P}_2 = \mathcal{P}_0^{\oplus n_0} \oplus \mathcal{P}_1^{\oplus n_1} \oplus \mathcal{P}_2^{\oplus n_2}$$

$$\chi_4(g) = (\chi_2(g))^2$$

$$n_0 = (\chi_0, \chi_4) = \frac{1}{6} (4 + 2) = 1$$

$$n_1 = (\chi_1, \chi_4) = \frac{1}{6} (4 + 2) = 1$$

$$n_2 = (\chi_2, \chi_4) = \frac{1}{6} (8 - 2) = 1$$

$$\Rightarrow \mathcal{P}_4 \cong \mathcal{P}_0 \oplus \mathcal{P}_1 \oplus \mathcal{P}_2$$