5. REPRESENTATIONS of S_N

- Show up in quantum desc of many particle systems
- Play an important role in the representation theory of SU(n).

$|S_n| = n! < \infty$ all complex repr. are fully decomp.

need to study irred. repres $\xleftrightarrow{1:1}$ conjugacy class

Recall

- conjugacy classes

$S_n \to \pi = (i_1^1, \dots i_{r_1}^1) \cdot (i_1^2, \dots i_{r_2}^2) \cdot \dots \cdot (i_1^n \dots i_{r_n}^n)$

$k_i :=$ # cycles of length i $\qquad \sum i k_i = n$

$(k_1, \dots k_2) \leftarrow$ cycle structure

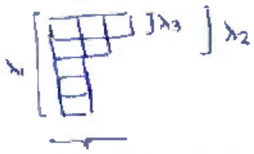
↳ determines the conjugacy class

- another parametrization by partitions : $k_i = \lambda_i - \lambda_{i+1}$

$\lambda_i :=$ # cycles of length $\geq i$ $\qquad \lambda_i = \sum_{j \geq i} k_j$

$\lambda_1 \geq \lambda_2 \geq \dots \qquad \sum \lambda_i = n$

pictorial repres. of partitions by means of Young diagrams



n boxes in colums of hight λ_i

$(k_1, k_2, \dots) \qquad (\lambda_1, \lambda_2, \dots$

$(2, 1, 1, 0 \dots) \sim (4, 2, 1, 0 \dots) \sim$ 

- Regular representation on the group algebra $\mathbb{C}[G]$

$V_{reg} = \mathbb{C}[G] = \{ \sum_{g \in G} \alpha_g \cdot g \mid \alpha_g \in \mathbb{C} \}$

$$\left.\begin{aligned} &\lambda (\sum \alpha_g \cdot g) = \sum (\lambda \alpha_g) \cdot g \\ &\sum \alpha_g \cdot g + \sum \beta_g \cdot g = \sum (\alpha_g + \beta_g) g \\ &\Big(\sum_{g \in G} \alpha_g\, g \Big) \cdot \Big(\sum_{h \in G} \beta_h \cdot h \Big) = \sum_{g, h \in G} (\alpha_g \cdot \beta_h)(g \cdot h) = \\ &\qquad = \sum_{g \in G} \Big(\sum_{h \in G} (\alpha_{gh^{-1}} \beta_h) \Big) g \end{aligned}\right\} \text{Group algebra structure}$$

$$\rho_{reg}(g)\left(\sum_h \alpha_h h\right) = \sum \alpha_h (gh) \overset{g\to}{=} \sum \alpha_{g^{-1}h}\, h$$

representation

$$\rho_{reg} = \rho_1^{\oplus a_1} \oplus \rho_2^{\oplus a_2} \oplus \dots \oplus \rho_k^{\oplus a_k} \qquad a_i = \dim(\rho_i)$$

$\to$ irreducibles' in V_{reg} are ideals in group algebra.

<u>Aim</u>: Construct irr. reprs. of S_n as subrepr. of ρ_{reg}!

① Start with a Young diagram:

$$\begin{array}{|c|c|c|}\hline & & \\ \hline & \\ \cline{1-2} \\ \cline{1-1} \\ \cline{1-1}\end{array} \longmapsto \begin{array}{|c|c|c|}\hline 1 & 5 & 7 \\ \hline 2 & 6 \\ \cline{1-2} 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} = T \quad \text{(Young tableau)}$$

diagram $\longmapsto$ Tableau

by filling in the numbers $1, \dots, n$, in such a way that numbers increase from left to right and top to bottom.

② $R(T) = \{\sigma \in S_n \mid \sigma \text{ leaves rows } (T) \text{ invariant}\}$

$C(T) = \{\sigma \in S_n \mid \sigma \text{ leaves colums } (T) \text{ invariant}\}$

groups of row- and colums- transf. of T.

$R(T) \cap C(T) = \{e\}$

<u>Example</u>: $T = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$ $\quad R(T) = \{e, (14)\}$

$C(T) = \{e, (12), (23), (13)(123), (132)\}$

③ $a_T := \sum_{\sigma \in R(T)} \sigma \in \mathbb{C}(S_n)$; $b_T := \sum_{\sigma \in C(T)} \operatorname{sign}(\sigma) \in \mathbb{C}(S_n)$

↳ symmetrizes the row transf ↳ antisymm. the colum transf.

Young symmetrizer: $c_T = a_T \cdot b_T$

<u>Example</u>: i) $T = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}$ $\quad c_T = a_T = \sum_{\sigma \in S_3} \sigma = 3!\, \Pi_0$

↖ projector on trivial reprs.

ii) $T = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$ $\quad c_T = b_T = \sum_{\sigma \in S_3} \operatorname{sign}(\sigma) \cdot \sigma = 3!\, \Pi_1$

↖ projector on sign reprs.

iii) $T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$ $\quad c_T = \underbrace{(e + (13))}_{a_T} \cdot (e - (12)) = e + (13) - (12) - (132) = 3\Pi_2$

④ $V_T := \overset{A}{\mathbb{C}[S_n]} \cdot c_T$ is a subrepres. of V_{reg}

$= \{a\, c\, c_T \mid a \in A\}$

Examples

i) $\boxed{1|2|3} = T$

$C(T) = \{e\}$

$R(T) = \{e, (1\,2), (2\,3), (1\,3), (1\,2\,3), (1\,3\,2)\} \equiv S_3$

$\mathbb{C}(S_3) \cdot c_T = \mathbb{C}(S_3) \cdot \sum_{\sigma \in S_3} \sigma = \mathbb{C} \cdot c_T \leftarrow$ 1-dim

$\left[\pi \sum_{\sigma \in S_3} \sigma = \sum_{\sigma \in S_3} (\pi\sigma) = \sum_{\sigma \in S_3} \sigma \right]$

$\rho_{reg}(\pi) c_T = \rho_{reg}(\pi) \sum_{\sigma \in S_3} \sigma = \sum_{\sigma \in S_3} (\pi\sigma) = \sum_{\sigma \in S_3} \sigma = c_T$

$\rightarrow$ trivial repr.!

ii) $T = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$ $\quad R(T) = e, \quad C(T) = S_3$

$V_T = \mathbb{C}(S_3)\, c_T = \mathbb{C}(S_3) \sum_{\sigma \in S_3} \operatorname{sign}(\sigma) \cdot \sigma = \mathbb{C} c_T$

$\rho_{reg}(\pi)\, c_T = \pi \sum_{\sigma \in S_3} \operatorname{sign}(\sigma)\, \sigma =$

$= \sum_{\sigma \in S_3} \operatorname{sign}(\sigma) \cdot (\pi\sigma) =$

$= \operatorname{sign}(\pi) \cdot \sum_{\sigma \in S_3} \operatorname{sign}(\pi\sigma)(\pi\sigma) =$

$= \operatorname{sign}(\pi) \cdot \sum_{\sigma \in S_3} \operatorname{sign}(\pi) \cdot \sigma =$

$= \operatorname{sign}(\pi) \cdot c_T$

$\hookrightarrow \rho_{reg}|_{V_T}$ is the sign repres.

iii) $T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$ $\quad c_T = e + (1\,3) - (1\,2) - (1\,2\,3)$

$V_T = \mathbb{C}[S_3] \cdot c_T$ is 2 dimensional, spanned by c_T and $(12)\, c_T$

$(1\,3)\, c_T = c_T$

$\rho_{reg}|_{V_T}$ is the irreducible 2d repres.

Fact: (i) $\rho_{reg}|_{V_T}$ is irreducible subrepresentation of ρ_{reg}.

(ii) $V_T \cong V_{T'}$ iff $\lambda(T) = \lambda(T')$

→ irreducible repres. only depends on underlying Young diagram.

→ there are as many copies of an irreducible representation corresponding to $(\lambda_1, \dots \lambda_n)$ as there are Young tableau built on this Young diagram.

Recall.

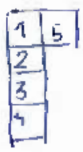
$$\rho_{reg} = \rho_1^{\oplus \dim(\rho_1)} \oplus \dots \oplus \rho_n^{\oplus \dim(\rho_n)}$$

→ irreducible represent. corresponding to a Young diagram has dimension the number of ways you can make a young tableau out of it.

Young diagram:

 ↦ irreducible repres. of S_n

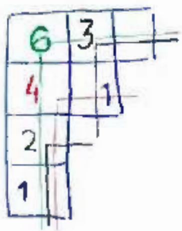
Young tableau



↦ subrepresentation of type  in ρ_{reg}.

- Hook-rule:

Formula for dimensions of irreducible representation of S_n.



Labels irreducible repres.

Hook numbers. draw a hook through a box and count the number of boxes met by the hook. h_i

$$\boxed{\dim(\rho_{\lambda}) = \frac{n!}{\prod h_i}}$$ Hook rule

Examples:

i) $\underbrace{\boxed{n}\boxed{n-1}\boxed{n-2}\cdots\boxed{1}}_{n\text{-boxes}}$ $\quad \dim(\rho_{\square\cdots\square}) = 1 \quad$ trivial repres.

$S_n \qquad d = \frac{n!}{n(n-1)\cdots 1} = 1$

ii) $\begin{array}{|c|}\hline n \\ \hline n-1 \\ \hline \vdots \\ \hline 1 \\ \hline\end{array}$ $\quad \dim(\rho_{\boxminus}) = 1 \quad$ sign-repres

$d = \frac{n!}{n(n-1)\cdots} = 1$

iii) $n-1 \left\{ \begin{array}{|c|c|}\hline n & 1 \\ \hline n-2 \\ \cline{1-1} \vdots \\ \cline{1-1} 1 \\ \cline{1-1}\end{array} \right.$ $\quad d = \frac{n!}{n(n-2)\cdots 1} = (n-1)$

Same for $\begin{array}{|c|c|c|c|}\hline \, & \, & \, & \cdots \\ \hline \, \\ \cline{1-1}\end{array}$

- Frobenius formula (for characters of S_n)
  - denote irreduc repres by partitions $(\lambda_1, \dots \lambda_n)$
  - denote conjugacy classes by the cycle structure $(k_1, \dots k_n)$

  Define polynomials:

$$P_{(k_1 \dots k_n)}(x_1 \dots x_n) = \prod_{i<j}(x_i - x_j)(x_1 + \dots + x_n)^{k_1} \cdot (x_1^2 + \dots + x_n^2)^{k_2} \cdot (x_1^3 + \dots + x_n^3)^{k_3} \cdots$$
$$\cdots (x_1^n + \dots x_n^n)^{k_n}$$

$$\underset{\substack{\| \\ (\lambda_1, \dots \lambda_n)}}{\chi_\lambda}(\underset{(k_1 \dots k_n)}{C_k}) = \left[P_{(k_1 \dots k_n)}(x_1 \dots x_n)\right]_{x_1^{\ell_1} \dots x_n^{\ell_n}}$$

coeff. of in

$$\ell_i = \lambda_i + n - i$$

Example

$G = S_3 \qquad P_{(1,1,0)}(x_1, x_2, x_3) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)(x_1 + x_2 + x_3)(x_1^2 + x_2^2 + x_3^2)$

$(\cdot\cdot)(\cdot)$

$\boxminus \quad \lambda = (3,0,0) \quad \ell = (5,1,0) \rightsquigarrow$ coeff. of $x_1^5 x_2$ in $P \qquad \chi_\lambda(C_k) = 1$

$\square\square\square \quad \lambda = (1,1,1) \quad \ell = (3,2,1) \rightsquigarrow$ coeff. of $x_1^3 x_2^2 \cdot x_3 \qquad \chi_\lambda(C_k) = -1$

$\lambda = (2,1,0) \quad \ell = (4,2,0) \rightsquigarrow$ coef. of $x_1^4 x_2^2 \qquad \chi_\lambda(C_k) = 0$

$\rightsquigarrow C_2$ - colum of characters table of $D_3 \equiv S_3$.