1. Abstract Groups

Lecture 2, Group Theory, Prof. Daniel Roggenkamp

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Definition: A *group* G:

$$G \times G \to G$$
$$(g_1, g_2) \mapsto g_1 g_2$$

s.t.

(i) *Associativity:* $(g_1 g_2) g_3 = g_1 (g_2 g_3)$
(ii) *Identity:* $\exists e \in G : e \cdot g = g \quad \forall g \in G$
(iii) *Inverse:* $\forall g \in G \; \exists g^{-1} : g^{-1} g = e$

In case $\forall g, h \in G : gh = hg$, then G is *Abelian* (commutative).

Remark:

(i) *right-inverses:*

$$g^{-1} g = e, \quad (g^{-1} g) g^{-1} = e g^{-1} = g^{-1} = g^{-1}(g g^{-1}) \quad \Rightarrow \quad \boxed{g g^{-1} = e}$$

(ii) *right-identity:*

$$g \cdot e = g(g^{-1} g) = (g g^{-1}) g = eg = g$$

(iii) *uniqueness of e:* Assume $\exists e' : e' g = g \; \forall g$

$$e' \cdot e = e = e'$$

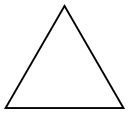
(iv) *uniqueness of inverse:* Assume that for $g \in G \; \exists h : hg = e$

$$h = h \cdot e = h(g g^{-1}) = (hg) g^{-1} = e g^{-1} = g^{-1}$$

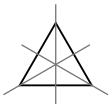
Examples:

(i) $(\mathbb{Z}, +)$ Abelian group
(ii) Cyclic groups $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$
(iii) Symmetric group S_n (permutations of n objects)
(iv) Dihedral group D_3 (symmetries of a regular triangle $\triangle$)

Example: Dihedral group D_3:

- D_3  Elements (6):

$$|D_3| = 6, \quad D_3 = \left\{ e_0 = R_0, R_1 = R\left(\frac{2\pi}{3}\right), R_2 = R\left(\frac{4\pi}{3}\right), \underbrace{S_0, S_1, S_2}_{\text{Reflections}} \right\}$$



$$\boxed{\begin{aligned} R_i R_j &= R_{i+j \bmod 3} \\ S_i^2 &= R_0 \\ R_i S_j &= S_{i+j \bmod 3} \end{aligned}}$$

these properties define the group

Using these relations: (these are consequences)

$$S_i R_j = S_{i-j},$$
$$S_i S_j = R_{i-j} \quad \text{(check)}$$

- D_n: symmetries of a regular n-gon

$$D_n = \{R_0, R_1, ..., R_{n-1}, S_0, ..., S_{n-1}\} \qquad (*)$$

So far, all examples are discrete. But now: continuous ones.

Lie groups:

(v) translation group $(\mathbb{R}^n, +)$, Abelian.

(vi) General linear group

$$\mathrm{GL}(n) := \{A \in \mathrm{Mat}(n, n; \mathbb{R}) \mid \det(A) \neq 0\}$$

(vii) (special) orthogonal group

$$(\mathrm{S})\mathrm{O}(n) := \{A \in \mathrm{Mat}(n, n; \mathbb{R}) \mid A^t A = \mathbb{1}_n\} \quad (\det(A) = 1)$$

(viii) Euclidean group (symmetries of an affine space)

$$E(n) := \{(a, A) \in \mathbb{R}^n \times O(n)\}, \quad (a, A)(a', A') = (a + Aa', AA')$$

(ix) (special) unitary group

$$(\mathrm{S})\mathrm{U}(n) := \left\{A \in \mathrm{Mat}(n, n; \mathbb{C}) \mid \overline{A}^t A = \mathbb{1}_n\right\} \quad (\det(A) = 1)$$

Def: A *group homomorphism* is a map

$$\varphi : G \to H \quad \text{s.t.} \quad \varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2).$$

- If φ is invertible: *group isomorphism.*
- $\mathrm{Hom}(G, H)$: set of homomorphisms.

Rem:

(i) $\varphi(e_G) = e_H$

$$[\varphi(g) = \varphi(g \cdot e_G) = \varphi(g) \cdot \varphi(e_G) \quad \Rightarrow \quad \varphi(e_G) = e_H]$$

(ii) $\varphi(g^{-1}) = (\varphi(g))^{-1}$

$$[\varphi(g^{-1})\varphi(g) = \varphi(g^{-1}g) = \varphi(e_G) = e_H \quad \Rightarrow \quad \varphi(g^{-1}) = (\varphi(g))^{-1}]$$

Examples:

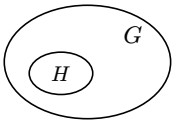
(i) $\mathbb{Z}_n \to D_n, \quad [a] \mapsto R_{a \bmod n}, \quad R_i \cdot R_j = R_{i+j}$

(ii) $\mathbb{Z}_n \to \mathrm{U}(1) = \{a \in \mathbb{C} \mid \overline{a}a = 1\}, \quad [a] \mapsto e^{\frac{2\pi i a}{n}}$

(iii) $\varphi : O(2) \to O(3), \quad A \mapsto \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array}\right)$

(iv) $\mathrm{U}(n) \to \mathrm{U}(1), \quad A \mapsto \det(A)$

Def: A *subgroup* $H \subset G$ is a subset which is itself a group w.r.t. multiplication in G.



Rem:

(i) H has to contain $e_G = e_H \overset{!}{\in} H$

(ii) $H \subset G$ (subgroup) $\Leftrightarrow \forall h_1, h_2 \in H : h_1 h_2^{-1} \in H$

(iii) Trivial subgroups: $\{e\} \subset G$, $G \subset G$

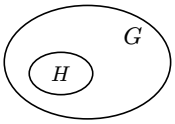
(iv) G Abelian $\Rightarrow H \subset G$ also Abelian.

Examples:

(i) $\mathbb{Z}_n \subset D_n$, $\mathbb{Z}_n = \{R_0, R_1, ..., R_{n-1}\}$
(ii) $\mathbb{Z}_n \subset \mathbb{Z}_{nm}$, $\mathbb{Z}_n \cong \{[0], [m], [2m], ..., [(n-1)m]\}$
(iii) $\mathrm{U}(1) \subset \mathrm{U}(n)$, $\mathrm{U}(1) \cong \left\{e^{i\varphi}\mathbb{1}_n \mid \varphi \in \mathbb{R}\right\}$
(iv) $\mathrm{SO}(n) \subset O(n)$, $\mathrm{SU}(n) \subset \mathrm{U}(n)$
(v) $O(n-1) \subset O(n)$, $\left(\begin{array}{c|c} A & \\ \hline & 1 \end{array}\right) \in O(n)$
(vi) $\forall g \in G : H = \Gamma_g = \left\{g^i \mid i \in \mathbb{Z}\right\}$ (Abelian)
(vii) center $Z(G) := \{h \in G \mid hg = gh \ \forall g \in G\}$

$$Z(O(n)) = \{\pm\mathbb{1}_n\} \cong \mathbb{Z}_2,$$
$$Z(D_{\text{even}}) = \left\{R_0, R_{n/2}\right\} \cong \mathbb{Z}_2.$$

Def: $H \subset G$: *left-coset* $g \cdot H = \{g \cdot h \mid h \in H\} \subset G$



Rem:

(i) gH is the rest class of the equivalence relation

$$a \sim b :\Leftrightarrow a^{-1}b \in H$$

[reflexive, symmetric, transitive]
(ii) g_1H and g_2H are either disjoint or identical:

$$g \in g_1H \cap g_2H \quad \Rightarrow \quad g_1h_1 = g = g_2h_2, \quad g_1 = g_2h_2h_1^{-1} \in g_2H$$

(iii) $g \cdot H$ is a subgroup $\Leftrightarrow g \in H$
(iv) $(gH) \mapsto (gH)^{-1} = Hg^{-1}$: left-coset $\leftrightarrow$ right coset

Examples:

(i) $G = (\mathbb{Z}, +)$, $H = m\mathbb{Z}$, cosets $m\mathbb{Z}, m\mathbb{Z}+1, m\mathbb{Z}+2, ..., m\mathbb{Z}+(m-1)$
(ii) $G = D_3$, $H = \mathbb{Z}_3$: $R_0\mathbb{Z}_3 = \{R_0, R_1, R_2\}$, $S_0\mathbb{Z}_3 = S_1\mathbb{Z}_3 = S_2\mathbb{Z}_3$

Def: $g_1, g_2 \in G$ are *conjugate*, $g_1 \sim g_2 :\Leftrightarrow \exists g \in G : g_1 = gg_2g^{-1}$.

This is an equivalence relation: rest classes

$$C_g = [g] = \left\{\tilde{g}g\tilde{g}^{-1} \mid \tilde{g} \in G\right\} \quad \text{conjugacy classes}$$

Example: D_n:

$$C_{R_i} = \{R_i, R_{-i}\}, \quad C_{S_i} = \left\{S_{i+2j} \mid j \in \mathbb{Z}\right\}$$

Def: A subgroup $H \subset G$ is *normal*, $H \lhd G$, if it is self-conjugate, i.e.

$$gHg^{-1} = H \quad \forall g \in G.$$

Remark:

(i) $H \lhd G$: $gH = Hg$ (left = right cosets)
(ii) H is a union of conjugacy classes
(iii) $N \subset H \subset G$, $N \lhd G \Rightarrow N \lhd H$,
but in general $N \lhd H \lhd G \not\Rightarrow N \lhd G$ WARNING!

Examples:

(i) center $Z(G)$
(ii) $\varphi : G \to G'$ homomorphism:

$$H \coloneqq \ker(\varphi) = \{g \in G \mid \varphi(g) = e_{G'}\} \lhd G$$

- subgroup: $\varphi(g) = e = \varphi(h)\ \forall g, h \in H$

$$\Rightarrow \varphi(g) \cdot \underbrace{\varphi(h^{-1})}_{e} = \varphi(gh^{-1}) \Rightarrow gh^{-1} \in H$$

- normal:

$$\varphi(ghg^{-1}) = \varphi(g)\underbrace{\varphi(h)}_{e}\varphi(g^{-1}) = \varphi(g)\varphi(g^{-1}) = \varphi(gg^{-1}) = e \Rightarrow ghg^{-1} \in H\ \forall g \in G$$

e.g. $\det : \mathrm{U}(n) \to \mathrm{U}(1)$ homomorphism,

$$\ker(\det) = \{A \in \mathrm{U}(n) \mid \det(A) = 1\} = \mathrm{SU}(n) \Rightarrow \mathrm{SU}(n) \lhd \mathrm{U}(n)$$

(iii) images of normal subgroups are normal in the image of the group:

$$H \lhd G \Rightarrow \varphi(H) \lhd \varphi(G) \subset G', \quad \varphi : G \to G'$$

Quotient group: If $H \lhd G$, then $G/H = \{gH \mid g \in G\}$ forms a group:

$$gHg'H = gg'\underbrace{(g'^{-1}Hg')}_{H}H = gg'H \cdot H = gg'H,$$

a well-defined multiplication map:
(i) associative
(ii) identity: H
(iii) inverse: $(gH)^{-1} = g^{-1}H$

Def: *Simple groups* G don't have non-trivial normal subgroups (building blocks).

Fact: $\varphi : G \to G'$, $H \coloneqq \ker(\varphi) \lhd G$

$$G/H = G/\ker(\varphi) \to G'$$
$$g\ker(\varphi) \mapsto \varphi(g)$$

$$f : G/H \xrightarrow{\cong} \varphi(G) \quad \text{isomorphism}$$

e.g. $\varphi = \det : \mathrm{U}(n) \to \mathrm{U}(1)$, $\ker(\varphi) = \mathrm{SU}(n)$:

$$\mathrm{U}(n)/\mathrm{SU}(n) \cong \mathrm{U}(1)$$

Product of groups:

Direct product $G_1 \times G_2$: for $(g_1, g_2), (g_1', g_2') \in G_1 \times G_2$,

$$(g_1, g_2) \cdot (g_1', g_2') = (g_1g_1', g_2g_2') \quad \to \text{satisfies group axioms.}$$

$$\pi_2 : G_1 \times G_2 \to G_2 \quad \text{(this is a group homomorphism)}$$
$$(g_1, g_2) \mapsto g_2$$

$$\ker(\pi_2) = G_1 \lhd G_1 \times G_2 \quad \text{(also true for } G_2\text{: } G_2 \lhd G_1 \times G_2\text{)}$$

Fact: $G = N_1 \times N_2 \Leftrightarrow$
(i) $N_1, N_2 \lhd G$
(ii) $N_1 \cap N_2 = \{e\}$
(iii) $G = N_1N_2$

Def: *semi-direct product*: N, H groups, $\theta : H \to \mathrm{Aut}(N) = \mathrm{Iso}(N)$,

$$N \rtimes_\theta H = N \times H \text{ as a set}, \quad (n,h), \quad n \in N, h \in H,$$

$$(n,h)\cdot(n',h') = (n \cdot \theta(h)(n'), hh').$$

If $\theta =$ identity, then this is the direct product.

$$e = (e_N, e_H), \quad (n,h)^{-1} = \big(\theta(h)^{-1}(n^{-1}), h^{-1}\big).$$

Check associativity:

$$((n,h)(n',h'))(n'',h'') = (n\theta(h)(n'), hh')(n'',h'') = (n\theta(h)(n') \underbrace{\theta(hh')}_{\theta(h)\circ\theta(h')}(n''), hh'h'')$$

$$\overset{\theta \text{ is hom.}}{=} (n\theta(h)(n'\theta(h')(n'')), hh'h'') = (n,h)((n',h')(n'',h'')).$$

Examples:

(i) $E(n) = \mathbb{R}^n \rtimes_\theta O(n)$,

$$\theta : O(n) \to \mathrm{Iso}(\mathbb{R}^n)$$
$$A \mapsto (a \in \mathbb{R}^n \mapsto A \cdot a)$$

(ii) $G = D_n$, $N = \mathbb{Z}_n \lhd D_n$, $H = \{R_0, S_0\} = \mathbb{Z}_2$:

$$\varphi : \mathbb{Z}_n \rtimes \mathbb{Z}_2 \to D_n$$
$$(R_i, R_0 = e) \mapsto R_i R_0 = R_i$$
$$(R_i, S_0) \mapsto R_i S_0 = S_i$$

$$\varphi(R_i,a)\varphi\big(R_j,b\big) = R_i a R_j b = R_i \underbrace{aR_ja^{-1}}_{R_j'} ab = \varphi\big(R_i\theta(a)\big(R_j\big), ab\big),$$

$$R_j' = aR_ja^{-1} = \theta(a)\big(R_j\big),$$

$$\theta : \mathbb{Z}_2 \to \mathrm{Iso}(\mathbb{Z}_n)$$
$$a \mapsto \big(R_i \mapsto aR_ia^{-1}\big)$$

22.04.2015

Next lecture: 13.05.2015. Problem set!!!

Summary of last lecture: Defined basic concepts of groups:

- groups
- subgroups
- homomorphisms
- conjugation
- normal groups, quotient groups
- cosets
- (semi-direct) products