3. (Linear) Representations

Lecture 4, Group Theory, Prof. Daniel Roggenkamp

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Def: A *(linear) representation* of a group G is a group homomorphism

$$\rho : G \to \boxed{\mathrm{GL}(V) = \{\text{invertible, linear transf. of } V\}}, \quad V : \text{vector space (repr. space)}.$$

$\mathrm{GL}(V)$ is the *general linear group* on V: the group of all automorphisms of V, i.e. the set of all bijective linear transformations $V \to V$ (isomorphisms from an object to itself) together with functional composition as group operation. Loosely speaking, the "symmetry group of the object".

Remark: If $\boxed{\dim(V) = n < \infty}$ (the *dimension of the representation*), then by choosing a basis, one can think of a representation as a collection of invertible $n \times n$ matrices $\rho(g)$, $g \in G$, s.t.

$$\rho(e) = \mathbb{1}_n, \quad \rho(g)\rho(g') = \rho(gg') \quad \forall g, g' \in G.$$

Examples

(i) $G = \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, representations on $V = \mathbb{C}$:

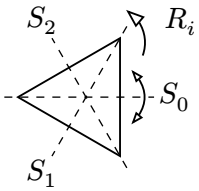
$$\mathrm{GL}(V) = \mathbb{C}^\times = \mathbb{C} \setminus \{0\} \quad \text{(complex plane without zero)} \quad \to \dim(V) = n = 1,$$

$$\rho([a]) = e^{\frac{2\pi i a}{n}}, \quad \rho([a])\rho([b]) = e^{\frac{2\pi i}{n}(a+b)} = \rho([a+b]).$$

(ii) $V = \mathbb{R}^2$:

$$\rho([a]) = \begin{pmatrix} \cos\left(\frac{2\pi}{n}a\right) & \sin\left(\frac{2\pi}{n}a\right) \\ -\sin\left(\frac{2\pi}{n}a\right) & \cos\left(\frac{2\pi}{n}a\right) \end{pmatrix}.$$

(iii) $G = D_n$, $V = \mathbb{R}^2$, $\dim(V) = n = 2$:



$$\rho(R_j) = \begin{pmatrix} \cos\left(\frac{2\pi j}{n}\right) & \sin\left(\frac{2\pi j}{n}\right) \\ -\sin\left(\frac{2\pi j}{n}\right) & \cos\left(\frac{2\pi j}{n}\right) \end{pmatrix}, \quad \rho(R_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\rho(S_0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho(S_j) = \begin{pmatrix} \cos\left(\frac{2\pi j}{n}\right) & -\sin\left(\frac{2\pi j}{n}\right) \\ -\sin\left(\frac{2\pi j}{n}\right) & -\cos\left(\frac{2\pi j}{n}\right) \end{pmatrix}.$$

$$R_iR_j = R_{i+j}$$
$$R_iS_j = S_{i+j}$$
$$S_iR_j = S_{i-j}$$
$$S_iS_j = R_{i-j}$$

Writing a for the angle $(2\pi a)/n$:

$$\rho(R_a)\rho(R_b) = \begin{pmatrix} \cos a & \sin a \\ -\sin a & \cos a \end{pmatrix}\begin{pmatrix} \cos b & \sin b \\ -\sin b & \cos b \end{pmatrix} = \begin{pmatrix} \cos a\cos b - \sin a\sin b & \cos a\sin b + \sin a\cos b \\ -\sin a\cos b - \cos a\sin b & -\sin a\sin b + \cos a\cos b \end{pmatrix}$$
$$= \begin{pmatrix} \cos(a+b) & \sin(a+b) \\ -\sin(a+b) & \cos(a+b) \end{pmatrix} = \rho(R_{a+b}),$$
$$\rho(R_a)\rho(S_b) = \begin{pmatrix} \cos a & \sin a \\ -\sin a & \cos a \end{pmatrix}\begin{pmatrix} \cos b & -\sin b \\ -\sin b & -\cos b \end{pmatrix} = \begin{pmatrix} \cos a\cos b - \sin a\sin b & -\cos a\sin b - \sin a\cos b \\ -\sin a\cos b - \cos a\sin b & \sin a\sin b - \cos a\cos b \end{pmatrix}$$
$$= \begin{pmatrix} \cos(a+b) & -\sin(a+b) \\ -\sin(a+b) & -\cos(a+b) \end{pmatrix} = \rho(S_{a+b}).$$

- Every group has a *trivial representation*:

$$\dim(V) = 1, \quad \rho(g) = 1 \quad \text{(maps everything to one)}, \quad \rho(g)\rho(g') = \rho(gg').$$

  $\rightsquigarrow$ This representation does not contain any information about the group.

- Representations containing all information about G (no information has been lost) are called *faithful*:

$$\rho : G \hookrightarrow \mathrm{GL}(V), \quad \text{which is injective.}$$

  This means: $\rho(g) = \rho(g') \to g = g'$. If it is *not* true: $\exists g \neq e : \rho(g) = \mathrm{id}_V$ (identity in V) $\leftarrow$ degenerate representation.

  Recall: $\ker(\rho) \lhd G \rightsquigarrow \rho$ descends to a faithful representation of $\boxed{G/\ker(\rho)}$.

  e.g. $G = S_n$
  (i) has a trivial one-dimensional representation, $\rho(g) = 1$,
  (ii) has a non-trivial one-dim. repr.: the *sign representation*

$$\rho(g) = (-1)^{\mathrm{length}(g)} \quad \big[\text{remember: } (-1)^{\mathrm{length}(g)} : S_n \to \mathbb{Z}_2\big], \quad \ker(\rho) = A_n \text{ (alternating group)}.$$

  ρ descends to a faithful representation of $S_n/A_n = \mathbb{Z}_2$.

- **Building representations:** Given representations[1]

$$\rho_1 : G \to \mathrm{GL}(V_1), \quad \rho_2 : G \to \mathrm{GL}(V_2).$$

  (i) *Direct sum*

$$\rho_1 \oplus \rho_2(g) = \rho_1(g) \oplus \rho_2(g) = \left(\begin{array}{c|c} \rho_1(g) & 0 \\ \hline 0 & \rho_2(g) \end{array}\right), \quad V_{\rho_1 \oplus \rho_2} = V_1 \oplus V_2,$$

$$\dim(\rho_1 \oplus \rho_2) = \dim(\rho_1) + \dim(\rho_2), \quad \dim(\rho) = \text{dim. of the representation vector space.}$$

  (ii) *Tensor product*: $V_{\rho_1 \otimes \rho_2} = V_{\rho_1} \otimes V_{\rho_2}$,

$$\rho_1 \otimes \rho_2(g) = \rho_1(g) \otimes \rho_2(g), \quad \dim(\rho_1 \otimes \rho_2) = \dim(\rho_1) \cdot \dim(\rho_2).$$

  (iii) *Dual representation*

$$\rho : G \to \mathrm{GL}(V), \quad \text{dual v.s. } V^* \coloneqq \mathrm{Hom}(V, \mathbb{K}),$$

$$\overline{\rho} : G \to \mathrm{GL}(V^*), \quad \overline{\rho}(g) = \left(\rho(g^{-1})\right)^* \quad \text{(dual map)}.$$

  About dual maps: $A : V \to W$, $A^* : W^* \to V^*$, $W^* = \mathrm{Hom}(W, \mathbb{K}) \ni w^*$, $A^*(w^*) = w^* \circ A : V \to \mathbb{K}$, $\in \mathrm{Hom}(V, \mathbb{K}) = V^*$. Prop. of dual maps: $A : V \to W$, $B : W \to U$, $(BA)^* = A^*B^*$.

  $\overline{\rho}$ is a representation:

$$\overline{\rho}(g)\overline{\rho}(h) = \left(\rho(g^{-1})\right)^* \left(\rho(h^{-1})\right)^* = \left(\rho(h^{-1})\rho(g^{-1})\right)^* = \left(\rho(h^{-1}g^{-1})\right)^* = \left(\rho((gh)^{-1})\right)^* = \overline{\rho}(gh).$$

- **Subrepresentations**

  Def: An *invariant subspace* W of a representation $\rho : G \to \mathrm{GL}(V)$ is a subspace $W \subset V$ s.t.

$$\rho(g)(W) \subset W \quad \Rightarrow \quad \rho(g)|_W \text{ is a subrepresentation.}$$

  $\rho|_W : G \to \mathrm{GL}(W)$ is a <u>representation</u> on its own.

[1] All the representations that we will consider are representations of a GL of a vector space.

Def: ρ is called *irreducible* if there are no invariant subspaces except $\{0\}$ and V. Otherwise (if it's not irreducible) it is called *reducible*. In matrix form (in a particular basis) this means

$$\rho(g) = \left(\begin{array}{c|c} \rho|_W\ (g) & * \\ \hline 0 & * \end{array}\right) \quad \text{(reducible representation)}.$$

- A repr. ρ is called *decomposable* if it is a direct sum $\rho = \rho_1 \oplus \rho_2$. In matrix form (in a particular basis):

$$\rho(g) = \left(\begin{array}{c|c} \rho_1(g) & 0 \\ \hline 0 & \rho_2(g) \end{array}\right).$$

  If it is not decomposable, it is called *indecomposable*.

- A repr. is called *fully decomposable* if it can be decomposed in a direct sum of irreducible repr.:

$$\rho = \rho_1 \oplus \cdots \oplus \rho_n, \quad \rho_i : \text{irreducible representations}.$$

Examples

(i) $\rho : \mathbb{Z} \to \mathrm{GL}(2)$,

$$\rho(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \quad \rho(n)\rho(m) = \rho(n+m): \quad \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & m+n \\ 0 & 1 \end{pmatrix}.$$

There is an invariant subspace (reducible): $\mathbb{R}^2 = \mathrm{span}(e_1, e_2)$, $W = \mathrm{span}(e_1)$ is invariant. But it is not decomposable, because $\rho(n)$ cannot be diagonalised. $\rho|_W$ is the trivial representation.

(ii) $G = S_3$, $V = \mathrm{span}\{e_1, e_2, e_3\}$, $\rho(\pi)(e_i) = e_{\pi(i)}$. In basis:

$$\rho((12)) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho((23)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Invariant subspace: $W = \mathrm{span}\{e_1 + e_2 + e_3\}$, because

$$\rho(\pi)(e_1 + e_2 + e_3) = e_{\pi(1)} + e_{\pi(2)} + e_{\pi(3)} = e_1 + e_2 + e_3.$$

$W^\perp = \mathrm{span}\{v_1 = e_1 - e_2, v_2 = e_2 - e_3\}$:

$$\rho((12))|_{W^\perp} : \begin{cases} v_1 \mapsto -v_1 \\ v_2 \mapsto v_1 + v_2 \end{cases} \quad \text{In matrix form:} \quad \rho((12)) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$\rho((23))|_{W^\perp} : \begin{cases} v_1 \mapsto v_1 + v_2 \\ v_2 \mapsto -v_2 \end{cases} \quad \text{In matrix form:} \quad \rho((23)) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}.$$

$\to \rho|_{W^\perp}$ is irreducible.

$$\rho = \underbrace{\rho|_W}_{\text{trivial}} \oplus \underbrace{\rho|_{W^\perp}}_{\text{2dim irred. repr.}}$$

$\rightsquigarrow$ Here we had a repr. that is $\to$ reducible, $\to$ fully decomposable.

Examples for building of representations:

$$\rho_r : \mathbb{Z}_n \to \mathrm{GL}(\mathbb{C}), \quad \rho_r([a]) = e^{\frac{2\pi i r a}{n}}.$$

(i) Direct sum:

$$\rho_r \oplus \rho_s([a]) = \begin{pmatrix} e^{\frac{2\pi i r a}{n}} & 0 \\ 0 & e^{\frac{2\pi i s a}{n}} \end{pmatrix}.$$

(ii) Tensor product: $\mathbb{C} \otimes \mathbb{C} = \mathbb{C} \rightsquigarrow$ we'll get again a 1-dim representation:

$$\rho_r \otimes \rho_s([a]) = \rho_r([a]) \otimes \rho_s([a]) = e^{\frac{2\pi i(r+s)a}{n}} = \rho_{r+s}([a]).$$

(iii) Dual representation: $\mathbb{C}^* = \mathbb{C}$,

$$\overline{\rho}_r([a]) = \rho_r([a]^{-1}) = \rho_r([-a]) = e^{-\frac{2\pi i r a}{n}} = \rho_{-r}([a]).$$

$$\overline{\rho}_r = \rho_{-r}, \quad \rho_r \otimes \rho_s = \rho_{r+s}.$$

Special representations compatible with some structures:

- **Unitary representation**

  Def: A *unitary representation* ρ is a repr. on a Hermitian vector space $(V, \langle\cdot,\cdot\rangle)$, $\langle a, b\rangle = \overline{\langle b, a\rangle}$, s.t.

  $$\rho : G \to \mathcal{U}(V) = \{T \in \mathrm{GL}(V) \mid \langle Ta, Tb\rangle = \langle a, b\rangle \ \forall a, b \in V\} \subset \mathrm{GL}(V),$$

  where the condition means $(T^*)^t T = T^\dagger T = \mathrm{id}_V$.

  Unitary representations are nice:

  Fact: Unitary repr. are fully decomposable.

  Proof. Assume $\rho : G \to U(V)$, $W \subset V$ is an invariant subspace $\Rightarrow W^\perp$ is an invariant subspace as well:

  $$W^\perp = \{v \in V \mid \langle v, w\rangle = 0 \ \forall w \in W\}.$$

  $$\forall v \in W^\perp, w \in W: \quad 0 = \langle v, w\rangle = \langle v, \rho(g)w\rangle = \langle \rho(g^{-1})v, w\rangle$$

  $\Rightarrow W^\perp$ is an invariant subspace as well $\Rightarrow \rho = \rho|_W \oplus \rho|_{W^\perp}$. Iteratively,

  $$\rho = \bigoplus_i \rho_i \quad (\rho_i : \text{irreducible representations}).$$

  □

  $\rightarrow$ The building blocks of unitary repres. are the irreducible repr.

- **Maps between representations**

  Def: An *intertwiner* between two representations

  $$\rho_1 : G \to \mathrm{GL}(V_1), \quad \rho_2 : G \to \mathrm{GL}(V_2)$$

  is a linear map $f : V_1 \to V_2$ s.t.

  $$\rho_2(g) \circ f = f \circ \rho_1(g) \quad \forall g \in G.$$

  $$\begin{array}{ccc} V_1 & \xrightarrow{\rho_1(g)} & V_1 \\ {\scriptstyle f}\downarrow & \circlearrowright & \downarrow{\scriptstyle f} \\ V_2 & \xrightarrow[\rho_2(g)]{} & V_2 \end{array} \qquad \text{only } \underline{\text{one}} \ f \text{ for every } g$$

If f is invertible $\Rightarrow$ equivalence of representations:

$$\rho_1(g) = f^{-1}\rho_2(g)f \quad \text{(change of basis transf.)},$$

(requires $\dim(V_1) = \dim(V_2)$).

Remark: $\mathrm{Im}(f)$ (image of f) and $\ker(f)$ are invariant subspaces.

Proof.

- $$\rho_2(g)\underbrace{f(v)}_{\in \mathrm{Im}(f)} \underset{\text{intertwiner}}{\overset{\text{def. of}}{=}} \underbrace{f\circ\rho_1(g)(v)}_{\in \mathrm{Im}(f)} \quad \Rightarrow \quad \mathrm{Im}(f) \text{ is an invariant subspace.}$$
- $f(v) = 0$, i.e. $v \in \ker(f)$:

$$\underbrace{\rho_2(g)f(v)}_{0} = f\circ\rho_1(g)v \quad \Rightarrow \quad \rho_1(g)v \in \ker(f) \quad \to \ker(f) \text{ is an invariant subspace.}$$

□

Schur's Lemma: If ρ_1 and ρ_2 are irreducible repr. of a group G and $f : V_1 \to V_2$ is an intertwiner, then:
(i) either $f \equiv 0$ or f is an isomorphism,
(ii) if $V_1 \cong V_2 \Rightarrow f = \lambda\,\mathrm{id}$, $\lambda \in \mathbb{C}$ (over $\mathbb{C}$).

Proof. (i) $f : V_1 \to V_2$ is an intertwiner.

$$\ker(f) \text{ is an invariant subspace} \overset{\text{irreducibility of } \rho_1}{\Rightarrow} \ker(f) = V_1\ (f \equiv 0) \quad \text{or} \quad \ker(f) = \{0\}\ (f \text{ injective}),$$

$$\mathrm{Im}(f) \text{ is an invariant subspace} \overset{\text{irreducibility of } \rho_2}{\Rightarrow} \mathrm{Im}(f) = \{0\}\ (f \equiv 0) \quad \text{or} \quad \mathrm{Im}(f) = V_2\ (f \text{ surjective}).$$

This shows (i).

(ii) Over $\mathbb{C}$ (we have already set $V = V_1 = V_2$): since $\mathbb{C}$ is algebraically closed, f has at least one eigenvalue λ.

$$f' = (f - \lambda \mathrm{id}_V) \text{ is still an intertwiner,}$$

but now $\ker(f') \neq \{0\}$, by irreducibility $\ker(f') = V$

$$\Rightarrow f' = (f - \lambda\ \mathrm{id}) \equiv 0.$$

This shows (ii). □

Remark: If two irreducible representations over $\mathbb{C}$ are equivalent as in (ii), then all the intertwiners are multiples of each other:

$$f_1, f_2 : V_1 \to V_2, \quad f_2 \neq 0 \quad \Rightarrow \quad \underbrace{f_2^{-1}f_1}_{\lambda\ \mathrm{id}} : V_1 \to V_1 \quad \Rightarrow \quad \boxed{f_1 = \lambda f_2}, \quad \lambda \in \mathbb{C}.$$

Note: this works over $\mathbb{C}$!

Aside: $\mathbb{R}$ is not algebraically closed

- $\nrightarrow$ not every intertwiner has an eigenvalue,
- $\nrightarrow$ the arguments above do not work,
- $\nrightarrow$ not all intertwiners over $\mathbb{R}$ are multiples of each other.

↳ This is the bad news. However, $\mathbb{R}$ still has enough structure: the intertwiners form an associative division algebra:

- can add them,

- can multiply them,
- if they are not zero, one can divide by them,
- since repr. by matrices, associative.

Division algebras:
- (associative) division algebra over $\mathbb{C}$: $\mathbb{C}$,
- division algebras over $\mathbb{R}$: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (quaternions).