4. Representations of Finite Groups

Lecture 5, Group Theory, Prof. Daniel Roggenkamp

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$|G| < \infty$, complex representations.

Fact: Any representation of G is fully decomposable.

Proof. $W \subset V$ (repr.) is an invariant subspace. Choose a projector $\pi_0 : V \to V$ onto W, i.e. $\mathrm{Im}(\pi_0) = W$. Define

$$\pi := \frac{1}{|G|} \sum_{g \in G} \rho(g) \pi_0 \rho(g^{-1}) : V \to V.$$

i) π is still a projector on W,
ii) π is an intertwiner.

From this it follows

$$V \overset{\pi \text{ is a projector}}{=} \underbrace{\mathrm{Im}(\pi)}_{W} \oplus \underbrace{\ker(\pi)}_{\substack{\text{is invariant subspace,} \\ \text{because } \pi \text{ is intertwiner}}}$$

i)

$$\pi^2 = \frac{1}{|G|^2} \sum_{g,k \in G} \rho(g) \pi_0 \rho(g^{-1}) \rho(k) \pi_0 \rho(k^{-1}) = \frac{1}{|G|} \sum_{k \in G} \rho(k) \pi_0 \rho(k^{-1}) = \pi,$$
$$\pi|_W = \mathrm{id}_W.$$

ii)

$$\pi \circ \rho(h) = \frac{1}{|G|} \sum_{g \in G} \rho(g) \pi_0 \overbrace{\rho(g^{-1}) \rho(h)}^{\rho(g^{-1}h)} \quad (g \mapsto hg)$$
$$= \frac{1}{|G|} \sum_{g \in G} \underbrace{\rho(hg)}_{\rho(h)\rho(g)} \pi_0 \rho(g^{-1}) = \rho(h) \pi.$$

□

Any representation ρ of a finite group decomposes into a sum of irreducible representations:

$$\rho = \rho_1^{\oplus n_1} \oplus \cdots \oplus \rho_k^{\oplus n_k}, \quad n_i \in \mathbb{N}_0.$$

Remark: Proof by averaging over the group → can be repeated for non-finite but compact groups, where instead of summing, one can integrate.

Decomposition

Start with a representation $\rho : G \to \mathrm{GL}(V)$,

$$\pi := \frac{1}{|G|} \sum_{g \in G} \rho(g) : V \to V.$$

- It is a projector on the invariant subspace

$$V^G := \{v \in V \mid \rho(g)v = v \; \forall g \in G\}.$$

(i)

$$\pi^2 = \frac{1}{|G|^2} \sum_{g,h \in G} \underbrace{\rho(g)\rho(h)}_{\rho(gh)} \quad (g \mapsto gh^{-1}) \quad = \frac{1}{|G|} \sum_{g \in G} \rho(g) = \pi.$$

(ii) $v \in V^G$: $\pi v = v \Rightarrow \pi|_{V^G} = \mathrm{id}_{V^G} \Rightarrow V^G \subset \mathrm{Im}(\pi)$.

$$\rho(g)\pi w = \rho(g) \frac{1}{|G|} \sum_{h \in G} \rho(h)w = \pi w$$

$\Rightarrow \mathrm{Im}(\pi) \subset V^G \Rightarrow V^G = \mathrm{Im}(\pi)$
$\to \pi$ projects on $V^G = \mathrm{Im}(\pi)$.

We want to know the dimension of V^G:

$$\dim\big(V^G\big) = \mathrm{Tr}_V(\pi) = \frac{1}{|G|} \sum_{g \in G} \underbrace{\mathrm{Tr}_V(\rho(g))}_{\chi_\rho(g) = \chi_V(g)} = \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g),$$

where $\chi_\rho(g) = \chi_V(g)$ is the *character* of the representation ρ, V.

Apply this:

$$\mathrm{Hom}(V_1, V_2) \cong V_1^* \otimes V_2 \quad \text{carries repr. } \overline{\rho}_1 \otimes \rho_2,$$

$$\overline{\rho}_1 \otimes \rho_2(g) : f \mapsto \rho_2(g) \circ f \circ \rho_1(g^{-1}),$$

$$\mathrm{Hom}(V_1, V_2)^G = \big\{f \in \mathrm{Hom}(V_1, V_2) \;\big|\; \rho_2(g) \circ f \circ \rho_1(g^{-1}) = f\big\} = \text{space of intertwiners.}$$

If ρ_1 and ρ_2 are irreducible, by Schur's lemma

$$\dim\Big(\mathrm{Hom}(V_1, V_2)^G\Big) = \begin{cases} 1, \rho_1 \cong \rho_2 \\ 0 \text{ otherwise.} \end{cases}$$

Then:

$$\begin{cases} 1, \rho_1 \cong \rho_2 \\ 0 \text{ otherwise} \end{cases} = \frac{1}{|G|} \sum_{g \in G} \chi_{\overline{\rho}_1 \otimes \rho_2}(g) = \frac{1}{|G|} \sum_{g \in G} \mathrm{Tr}_{V_1^* \otimes V_2}(\overline{\rho}_1(g) \otimes \rho_2(g)) \quad (\overline{\rho}_1(g) = \rho_1(g^{-1}) \text{ on } V_1)$$

$$= \frac{1}{|G|} \sum_{g \in G} \mathrm{Tr}_{V_1^*}(\overline{\rho}_1(g)) \cdot \underbrace{\mathrm{Tr}_{V_2}(\rho_2(g))}_{\chi_2(g)} = \frac{1}{|G|} \sum_{g \in G} \chi_1(g^{-1})\chi_2(g).$$

$$\frac{1}{|G|} \sum_{g \in G} \chi_i(g^{-1}) \cdot \chi_j(g) = \delta_{ij} \quad (\text{for } i, j \text{ running over irreducible representations}).$$

$$V = V_1^{\oplus n_1} \oplus \cdots \oplus V_k^{\oplus n_k}, \quad \mathrm{Hom}(V_i, V) = \bigoplus_a \mathrm{Hom}(V_i, V_a)^{\oplus n_a},$$

$$\boxed{n_i = \mathrm{Tr}_{\mathrm{Hom}(V_i, V)}(\pi_i) = \frac{1}{|G|} \sum_{g \in G} \chi_{\rho_i}(g^{-1})\chi_\rho(g)}$$

From characters one can obtain the decomposition of V.

Can do more: identify the respective subrepresentation.

Next step: how to localize a subspace V_i in V.

Projector on $V_1^{\oplus n_1}$

$$\pi_{\rho_1} := \frac{\overbrace{\dim(V_1)}^{d_1}}{|G|} \sum_{g\in G} \chi_{\rho_1}(g^{-1})\rho(g) : V \to V.$$

i) π_{ρ_1} is an intertwiner:

$$\pi_{\rho_1} \circ \rho(h) = \frac{d_1}{|G|} \sum_{g\in G} \chi_{\rho_1}(g^{-1})\rho(gh) = \frac{d_1}{|G|} \sum_{g\in G} \underbrace{\chi_{\rho_1}(hg^{-1}h^{-1})}_{\chi_{\rho_1}(g^{-1})}\rho(hg) = \rho(h)\pi_{\rho_1},$$

by cyclicity of the trace:

$\chi_{\rho_1}(hg^{-1}h^{-1}) = \mathrm{Tr}(\rho_1(hg^{-1}h^{-1})) = \mathrm{Tr}(\rho_1(h)\rho_1(g^{-1})\rho_1(h^{-1})) = \mathrm{Tr}(\rho_1(h^{-1})\rho_1(h)\rho_1(g^{-1})) = \mathrm{Tr}(\rho_1(g^{-1}))$.

ii) π_ρ is a projector:

$$\begin{aligned}\pi_\rho^2 &= \frac{d_1^2}{|G|^2} \sum_{g,h\in G} \chi_{\rho_1}(g^{-1})\chi_{\rho_1}(h^{-1})\rho(gh) \quad (g \mapsto gh^{-1}) \\ &= \frac{d_1^2}{|G|^2} \sum_{g,h} \chi_{\rho_1}(hg^{-1})\chi_{\rho_1}(h^{-1})\rho(g) = \frac{d_1}{|G|} \sum_{g\in G} \chi_{\rho_1}(g^{-1})\rho(g) = \pi_{\rho_1},\end{aligned}$$

where we used Ex. 3c) from problem sheet 2:

$$\sum_h \chi_{\rho_i}(hg^{-1})\chi_{\rho_j}(h^{-1}) = \chi_{\rho_i}(g^{-1})\frac{|G|}{d_i}\delta_{ij}.$$

iii) π_{ρ_1} projects onto $V_1^{\oplus n_1} \subset V$. Look at the representation

$$\mathrm{Im}\big(\pi_{\rho_1}\big) = V_1^{\oplus n_1'} \oplus \cdots \oplus V_k^{\oplus n_k'}.$$

Want to show that $n_1' = n_1$:

$$n_i' = \frac{1}{|G|} \sum_{g\in G} \chi_{\rho_i}(g^{-1})\chi_{\mathrm{Im}(\pi_{\rho_1})}(g),$$

$$\chi_{\mathrm{Im}(\pi_{\rho_1})}(g) = \mathrm{Tr}_{\mathrm{Im}(\pi_{\rho_1})}\rho(g) = \mathrm{Tr}_V\big(\pi_{\rho_1}\rho(g)\big) = \frac{d_1}{|G|} \sum_{h\in G} \chi_{\rho_1}(h^{-1}) \underbrace{\mathrm{Tr}_V(\rho(h)\rho(g))}_{\mathrm{Tr}_V(\rho(hg))=\chi_V(hg)} = \frac{d_1}{|G|} \sum_{h\in G} \chi_{\rho_1}(h^{-1})\chi_V(hg),$$

$$\begin{aligned}n_i' &= \frac{d_1}{|G|^2} \sum_{g,h} \chi_{\rho_i}(g^{-1})\chi_{\rho_1}(h^{-1})\chi_V(hg) \quad (g \mapsto h^{-1}g) \\ &= \frac{d_1}{|G|^2} \sum_{g,h} \chi_{\rho_i}(g^{-1}h)\chi_{\rho_1}(h^{-1})\chi_V(g) \overset{\text{Ex. } 3}{=} \delta_{i,1} \underbrace{\frac{1}{|G|} \sum_{g\in G} \chi_{\rho_i}(g^{-1})\chi_V(g)}_{n_i} = \delta_{i,1}n_i\end{aligned}$$

$\to \pi_{\rho_1}$ projects onto $V_1^{\oplus n_1} \subset V$.

Characters of irreducible representations

$$\chi_\rho(g) = \mathrm{Tr}(\rho(g)).$$

Properties

i) $\chi_{\rho_1\oplus\rho_2}(g) = \chi_{\rho_1}(g) + \chi_{\rho_2}(g)$,

ii) $\chi_{\rho_1\otimes\rho_2}(g) = \chi_{\rho_1}(g)\cdot\chi_{\rho_2}(g)$,

iii) $\chi_\rho(e) = \dim(\rho)$ (dimension of the representation space),

iv) $\chi_\rho(g^{-1}) = \overline{\chi_\rho(g)}$,

v) $\chi_\rho(h^{-1}gh) = \chi_\rho(g)$ by cyclicity of the trace: χ_ρ only depends on the conjugacy class of g!

$\to \chi_\rho$ is a class function.

Space of class functions:

$$\mathbb{C}^{\text{class}}(G) := \{f : G \to \mathbb{C} \mid f(h^{-1}gh) = f(g)\}$$

is a vector space, with

$$\dim\big(\mathbb{C}^{\text{class}}(G)\big) = \#\text{ conjugacy classes in } G.$$

On this space define a Hermitian form

$$(\alpha,\beta) := \frac{1}{|G|}\sum_{g\in G}\overline{\alpha(g)}\beta(g).$$

$$\chi_i := \chi_{\rho_i} \in \mathbb{C}^{\text{class}}(G) \quad (\rho_i \text{ irreducible}), \quad \big(\chi_i,\chi_j\big) = \delta_{i,j} \quad \text{orthonormal set}$$

$\to$ # irred. repres. $\le$ # conjugacy classes.

Want to show: χ_i are an ON basis of $\mathbb{C}^{\text{class}}(G)$
- χ_i are linearly independent
- $\to$ characterize representations completely.

Proof. We have already seen that the χ_i are an ON set in $\mathbb{C}^{\text{class}}(G)$. It is a basis if $\big(\alpha,\chi_j\big) = 0\ \forall j \Rightarrow \alpha = 0$.

Assume $(\alpha,\chi_i) = 0\ \forall i$.

$$\varphi_\alpha^{\rho_i} := \sum_g \overline{\alpha(g)}\rho_i(g) : V_i \to V_i.$$

This is an intertwiner:

$$\begin{aligned}\varphi_\alpha^{\rho_i}\rho_i(h) &= \sum_g \overline{\alpha(g)}\rho_i(gh) \quad (g\mapsto hgh^{-1})\\ &= \sum_g \overline{\alpha(hgh^{-1})}\rho_i(hg) = \rho_i(h)\sum_g \underbrace{\overline{\alpha(hgh^{-1})}}_{\overline{\alpha(g)}}\rho_i(g) = \rho_i(h)\varphi_\alpha^{\rho_i} \quad \to \text{intertwiner.}\end{aligned}$$

V_i irreducible, by Schur's lemma: $\varphi_\alpha^{\rho_i} = \lambda_i \mathrm{id}_{V_i}$,

$$\lambda_i = \frac{\mathrm{Tr}(\varphi_\alpha^{\rho_i})}{\dim(V_i)} = \frac{1}{d_i}\sum_g \overline{\alpha(g)}\chi_{\rho_i}(g) = \frac{|G|}{d_i}\big(\alpha,\chi_{\rho_i}\big) = 0 \quad\Rightarrow\quad \boxed{\varphi_\alpha^{\rho_i}\equiv 0} \quad \forall i$$

$$\Rightarrow \sum_g \overline{\alpha(g)}\rho_i(g) = 0 \quad \forall i$$

$$\Rightarrow \sum_g \overline{\alpha(g)}\rho(g) = 0 \quad \forall \text{ repres. } \rho.$$

Find a representation ρ such that all $\rho(g_i)$ are linearly independent $\Rightarrow \overline{\alpha(g)} = 0\ \forall g$.

A representation for which this holds is the so-called regular representation:

$$V = \mathbb{C}[G] = \left\{ \sum_i \alpha_i g_i \;\middle|\; \alpha_i \in \mathbb{C} \right\} \quad \text{(group algebra of } G\text{)},$$

$$\rho(g)\Big(\sum \alpha_i g_i\Big) = \sum \alpha_i (g g_i),$$

$$\sum \alpha_i \rho(g_i) g = \sum \alpha_i (g_i g) = 0 \quad \text{only if } \alpha_i = 0 \;\forall i$$

$\to \rho(g_i)$ are linearly independent. □

$\to \chi_i$ is an orthonormal basis of $\mathbb{C}^{\text{class}}(G)$, $\dim \mathbb{C}^{\text{class}}(G) = \#$ conj. classes of G

$\to$ as many irreducible representations as there are conjugacy classes

$\to$ Characters determine repr.

Character table

G	C_1	C_2	$\cdots$	C_k
ρ_1	$\chi_1(C_1) = d_1$	$\chi_1(C_2)$		
$\vdots$				
ρ_k	$\chi_k(C_1) = d_k$	$\cdots$		$\chi_k(C_k)$

C_1: conjugacy class of the identity,
$k = \#$ conj. classes of G,

$\boxed{\chi_{ia} \coloneqq \chi_i(C_a)}$

$$\delta_{ij} = \frac{1}{|G|} \sum_g \overline{\chi_i(g)} \chi_j(g) = \sum_a \overline{\chi_i(C_a)} \chi_j(C_a) \frac{|C_a|}{|G|} \quad \text{(orthogonality of rows)}.$$

With

$$D_{ab} \coloneqq \frac{|C_a|}{|G|} \delta_{ab} : \quad X D \overline{X}^T = \mathbb{1}_k \quad \Rightarrow \quad \big(\overline{X}^T\big) X = D^{-1}$$

$$\boxed{\Rightarrow \sum_i \overline{\chi_i(C_a)} \cdot \chi_i(C_b) = \delta_{ab} \frac{|G|}{|C_a|} \quad \text{(orthogonality of columns)}.}$$

This constrains the character table.

Another constraint from the regular representation:

$$\rho^{\text{reg}}(g)\Big(\sum \alpha_i g_i\Big) = \sum \alpha_i (g g_i),$$

$$\chi_{\text{reg}}(g) = \text{Tr}_{\mathbb{C}[G]}(\rho^{\text{reg}}(g)) = \#\{i \mid g \cdot g_i = g_i\} = \begin{cases} |G|, g = e \\ 0 \quad \text{otherwise} \end{cases}$$

where the trace is taken in the basis g_i, $\rho(g) g_i = (g \cdot g_i)$.

$$\rho^{\text{reg}} = \rho_1^{\oplus n_1} \oplus \cdots \oplus \rho_k^{\oplus n_k},$$

$$n_i = \big(\chi_{\rho_i}, \chi_{\text{reg}}\big) = \frac{1}{|G|} \sum_{g \in G} \chi_{\rho_i}(g^{-1}) \chi_{\rho^{\text{reg}}}(g) = \frac{1}{|G|} \dim(\rho_i) \cdot |G| = \dim(\rho_i)$$

$$\Rightarrow \rho^{\text{reg}} = \rho_1^{\oplus d_1} \oplus \cdots \oplus \rho_k^{\oplus d_k}, \quad d_i = \dim(\rho_i),$$

$$\dim(\rho^{\text{reg}}) = \dim(\mathbb{C}[G]) = |G| = \sum_i d_i^2 \quad \Rightarrow \quad \boxed{|G| = \sum d_i^2}$$

Example: $D_3 \cong S_3$, $|D_3| = 6$. Conjugacy classes:

$(\cdot)(\cdot)(\cdot)$  $C_0 = C(R_0) = \{R_0\} = \{e\}$

$(\cdot\,\cdot)(\cdot)$  $C_1 = C(S_0) = \{S_0, S_1, S_2\}$

$(\cdot\,\cdot\,\cdot)$  $C_2 = C(R_1) = \{R_1, R_2\}$

$\to$ 3 irreducible representations.

D_3/S_3	C_0	C_1	C_2
$\lvert C_a\rvert$	1	3	2
ρ^{triv}	1	1	1
ρ^{sgn}	1	-1	1
ρ_2 $(*)$	2	$\alpha = 0$	$\beta = -1$
ρ^P $(**)$	3	1	0

$(*)$ $|G| = \sum d_i^2$: $\quad 6 = 1 + 1 + d_2^2 \to d_2 = 2$.

Obtain α and β using orthogonality:

$$0 = (\chi_{\text{triv}}, \chi_2) = \frac{1}{6}(2 \cdot 1 + 3 \cdot \alpha + 2 \cdot \beta),$$

$$0 = \big(\chi_{\text{sgn}}, \chi_2\big) = \frac{1}{6}(2 - 3 \cdot \alpha + 2\beta) \quad \Rightarrow \quad \alpha = 0, \quad \beta = -1.$$

Look at

$$\rho(S_0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho(R_i) = \begin{pmatrix} \cos\big(\frac{2\pi}{3}\, i\big) & -\sin\big(\frac{2\pi i}{3}\big) \\ \sin\big(\frac{2\pi}{3}\, i\big) & \cos\big(\frac{2\pi i}{3}\big) \end{pmatrix},$$

$$\chi_\rho(C_0) = 2, \quad \chi_\rho(C_1) = 0, \quad \chi_\rho(C_2) = 2\cos\left(\frac{2\pi}{3}\right) = -1.$$

$\rho \cong \rho_2$ because the characters agree.

$(**)$ Permutation representation:

$$\rho^P(e) = \mathbb{1}_3, \quad \rho^P(R_1) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho^P(S_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\chi_P(C_0) = 3, \quad \chi_P(C_1) = 1, \quad \chi_P(C_2) = 0,$$

$$\rho^P = \rho_{\text{triv}}^{\oplus n_1} \oplus \rho_{\text{sign}}^{\oplus n_2} \oplus \rho_2^{\oplus n_3},$$

$$n_1 = (\chi_{\text{triv}}, \chi_3) = \frac{1}{6}(3 \cdot 1 + 3 \cdot 1) = 1,$$

$$n_2 = \big(\chi_{\text{sign}}, \chi_3\big) = \frac{1}{6}(3 \cdot 1 - 3 \cdot 1) = 0,$$

$$n_3 = (\chi_2, \chi_3) = \frac{1}{6}(3 \cdot 2) = 1,$$

where $\chi_3 = \chi_P$:

$$\boxed{\rho^P \cong \rho_{\text{triv}} \oplus \rho_2}$$

Projectors:

$$\pi_{\text{triv}} = \frac{1}{6}\sum_g \overline{\chi_{\text{triv}}(g)}\rho^P(g) = \frac{1}{3}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix} \quad \text{projector on } \mathbb{C}(e_1+e_2+e_3),$$

$$\pi_2 = \frac{\overbrace{2}^{\text{dimensionality}}}{6}\left(2\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}\right) = \frac{1}{3}\begin{pmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{pmatrix}$$

$\to$ projector on the space $\text{span}\{e_1 - e_2, e_2 - e_3\}$.

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Summary from last lecture:

- Complex representations of finite groups are fully decomposable:

$$\rho = \bigoplus_i \underbrace{\rho_i^{\oplus n_i}}_{\text{irreducible repres.}} = \underbrace{\rho_1 \oplus \cdots \oplus \rho_1}_{n_1} \oplus \underbrace{\rho_2 \oplus \cdots \oplus \rho_2}_{n_2} \oplus \cdots$$

- As many irred. reprs. as there are conjugacy classes in G.
- Repres. determined by characters: $\chi_\rho(g) = \text{Tr}_{V_\rho}(\rho(g))$, $\left(\chi_i = \chi_{\rho_i}\right)$, characters of irr. reps. are an ON basis of class functions.
- $$n_i = (\chi_i, \chi_\rho) = \frac{1}{|G|}\sum_{g\in G} \underbrace{\chi_i(g^{-1})}_{\overline{\chi_i(g)}}\chi_\rho(g).$$
- Projector

$$\pi_i = \frac{d_i}{|G|}\sum_{g\in G}\chi_i(g^{-1})\rho(g) : V \to V \text{ projects onto } V_i^{\oplus n_i} \subset V, \quad d_i \equiv \dim(\rho_i).$$

- Character table:

G	$C_1 = [e]$ $\lvert C_1\rvert = 1$	C_2 $\lvert C_2\rvert$	...	C_k $\lvert C_k\rvert$
ρ_1	$d_1 = \chi_1(C_1)$		$\cdots$	$\chi_1(C_k)$
$\vdots$	$\vdots$			
ρ_k	$d_k = \chi_k(C_1)$		$\cdots$	$\chi_k(C_k)$

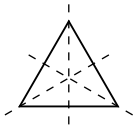
$$\sum_a \frac{|C_a|}{|G|}\overline{\chi_i(C_a)}\chi_j(C_a) = \delta_{i,j},$$

$$\sum_i \frac{|C_a|}{|G|}\overline{\chi_i(C_a)}\chi_i(C_b) = \delta_{ab},$$

$$\sum_i d_i^2 = |G|.$$

Example: $D_3 = \{R_0, R_1, R_2, S_0, S_1, S_2\}$, $R_0 = e$.

$$C_1 = C(R_0) = \{R_0\},$$
$$C_2 = C(S_0) = \{S_0, S_1, S_2\},$$
$$C_3 = C(R_1) = \{R_1, R_2\}.$$



D_3	C_1	C_2	C_3
	1	3	2
ρ_0	1	1	1
ρ_1	1	-1	1
ρ_2	2	0	-1
ρ_3	3 Tr $\rho_3(R_0)$	1 Tr $\rho_3(S_0)$	0 Tr $\rho_3(R_1)$

ρ_0: trivial repr., $V = \mathbb{C}$, $\rho_0(g) = 1$, $\chi_0(C_1) = 1$.

ρ_1: sign repr., $V = \mathbb{C}$,

$$\rho_1 : \begin{cases} R_i \mapsto 1 \\ S_i \mapsto -1 \end{cases} \quad \chi_1(C_1) = 1, \quad \chi_1(C_3) = 1, \quad \chi_1(C_2) = -1.$$

ρ_2: 2d rep.

$$\rho_2(R_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho_2(R_1) = \begin{pmatrix} \cos(\frac{2\pi}{3}) & \sin(\frac{2\pi}{3}) \\ -\sin(\frac{2\pi}{3}) & \cos(\frac{2\pi}{3}) \end{pmatrix}, \quad \rho_2(R_2) = \cdots,$$

$$\rho_2(S_0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho_2(S_1) = \cdots$$

Permutation representation: $V = \text{span}\{e_1, e_2, e_3\}$,

$$R_0 : \begin{cases} e_1 \mapsto e_1 \\ e_2 \mapsto e_2 \\ e_3 \mapsto e_3 \end{cases} \quad R_1 : \begin{cases} e_1 \mapsto e_2 \\ e_2 \mapsto e_3 \\ e_3 \mapsto e_1 \end{cases} \quad R_2 : \begin{cases} e_1 \mapsto e_3 \\ e_2 \mapsto e_1 \\ e_3 \mapsto e_2 \end{cases} \quad S_0 : \begin{cases} e_1 \mapsto e_1 \\ e_2 \mapsto e_3 \\ e_3 \mapsto e_2 \end{cases} \quad \cdots$$

$$\rho_3(R_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho_3(S_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\rho_3(R_2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \rho_3(S_1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\rho_3(R_1) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho_3(S_2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$\rho_3 = \rho_0^{\oplus n_0} \oplus \rho_1^{\oplus n_1} \oplus \rho_2^{\oplus n_2},$$

$$n_0 = (\chi_0, \chi_3) = \frac{1}{6}(3 \cdot 1 \cdot 1 + 1 \cdot 1 \cdot 3) = 1,$$
$$n_1 = (\chi_1, \chi_3) = \frac{1}{6}(3 - 3) = 0,$$
$$n_2 = (\chi_2, \chi_3) = \frac{1}{6} \cdot 6 = 1,$$

$$\boxed{\rho_3 \cong \rho_0 \oplus \rho_2}$$

Projector:

$$\pi_0 = \frac{1}{6}\sum_{g\in G}\overline{\chi_0(g)}\rho_3(g) = \frac{1}{3}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix} \quad \rightsquigarrow \text{ projection on } \operatorname{span}\{e_1+e_2+e_3\},$$

$$\pi_2 = \frac{2}{6}\sum_{g\in G}\overline{\chi_2(g)}\rho_3(g) = \frac{1}{3}\left(2\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}\right) = \frac{1}{3}\begin{pmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{pmatrix}$$

$\rightsquigarrow$ projection on $V_2 = \operatorname{span}\{e_1-e_2, e_2-e_3\}$,

with $V_0 \subset V_3 = \operatorname{span}(e_1,e_2,e_3)$, $V_0 = \mathbb{C}(e_1+e_2+e_3)$.

$$\rho_4 = \rho_2\otimes\rho_2 = \rho_0^{\oplus n_0}\oplus\rho_1^{\oplus n_1}\oplus\rho_2^{\oplus n_2}, \quad \chi_4(g) = (\chi_2(g))^2,$$

$$\left.\begin{aligned} n_0 &= (\chi_0,\chi_4) = \tfrac{1}{6}(4+2) = 1\\ n_1 &= (\chi_1,\chi_4) = \tfrac{1}{6}(4+2) = 1\\ n_2 &= (\chi_2,\chi_4) = \tfrac{1}{6}(8-2) = 1\end{aligned}\right\} \quad\Rightarrow\quad \rho_4 \cong \rho_0\oplus\rho_1\oplus\rho_2.$$