5. Representations of S_n

Lecture 6, Group Theory, Prof. Daniel Roggenkamp

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- Show up in the quantum description of many-particle systems.
- Play an important role in the representation theory of SU(n).

$|S_n| = n! < \infty \to$ all complex repr. are fully decomposable.

$$\text{need to study irred. repres.} \quad \overset{1:1}{\longleftrightarrow} \quad \text{conjugacy classes}$$

Recall

- Conjugacy classes:

$$S_n \ni \pi = \left(i_1^1, ..., i_{r_1}^1\right) \cdot \left(i_1^2, ..., i_{r_2}^2\right) \cdots \left(i_1^m ... i_{r_m}^m\right),$$

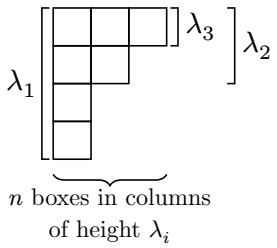
$$k_i := \#\text{ cycles of length } i, \quad \sum_i i k_i = n,$$

  $(k_1, ..., k_n) \leftarrow$ cycle structure, $\hookrightarrow$ determines the conjugacy class.

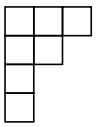
- Another parametrization by partitions:

$$\lambda_i := \#\text{ cycles of length} \geq i, \quad \lambda_i = \sum_{j \geq i} k_j, \quad k_i = \lambda_i - \lambda_{i+1}, \quad \lambda_1 \geq \lambda_2 \geq ..., \quad \sum \lambda_i = n.$$

  Pictorial representation of partitions by means of *Young diagrams*:



$$\begin{matrix}(k_1, k_2, ...) \\ (2, 1, 1, 0, ...)\end{matrix} \sim \begin{matrix}(\lambda_1, \lambda_2, ...) \\ (4, 2, 1, 0, ...)\end{matrix} \sim$$



- Regular representation on the group algebra $\mathbb{C}[G]$:

$$V_{\text{reg}} = \mathbb{C}[G] = \left\{ \sum_{g \in G} \alpha_g \cdot g \;\middle|\; \alpha_g \in \mathbb{C} \right\},$$

  with the group algebra structure

$$\lambda\left(\sum \alpha_g \cdot g\right) = \sum \left(\lambda \alpha_g\right) \cdot g,$$

$$\sum \alpha_g \cdot g + \sum \beta_g \cdot g = \sum \left(\alpha_g + \beta_g\right) g,$$

$$\left(\sum_{g \in G} \alpha_g g\right) \cdot \left(\sum_{h \in G} \beta_h h\right) = \sum_{g,h \in G} (\alpha_g \cdot \beta_h)(g \cdot h) = \sum_{g \in G} \left(\sum_{h \in G} \alpha_{gh^{-1}} \beta_h\right) g,$$

  and the representation

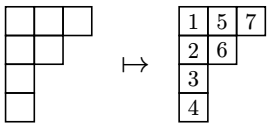
$$\rho_{\text{reg}}(g) \left(\sum_h \alpha_h h\right) = \sum \alpha_h (gh) \overset{h \to g^{-1}h}{=} \sum \alpha_{g^{-1}h} h,$$

$$\rho_{\text{reg}} = \rho_1^{\oplus d_1} \oplus \rho_2^{\oplus d_2} \oplus \cdots \oplus \rho_k^{\oplus d_k}, \quad d_i = \dim(\rho_i)$$

$\rightarrow$ 'irreducibles' in V_{reg} are ideals in the group algebra.

Aim: Construct irr. reprs. of S_n as subrepr. of ρ_{reg}!

① Start with a Young diagram:



$\mapsto$

$= T$ (Young tableau)

diagram $\mapsto$ tableau, by filling in the numbers $1, ..., n$ in such a way that numbers increase from left to right and top to bottom.

② Groups of row and column transformations of T:

$$R(T) = \{\sigma \in S_n \mid \sigma \text{ leaves rows}(T) \text{ invariant}\},$$
$$C(T) = \{\sigma \in S_n \mid \sigma \text{ leaves columns}(T) \text{ invariant}\},$$
$$R(T) \cap C(T) = \{e\}.$$

Example:

$$T = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array} \quad R(T) = \{e, (14)\},$$
$$C(T) = \{e, (12), (23), (13), (123), (132)\}.$$

③

$$a_T := \sum_{\sigma \in R(T)} \sigma \in \mathbb{C}[S_n] \quad \text{(symmetrizes the row transf.)},$$
$$b_T := \sum_{\sigma \in C(T)} \text{sgn}(\sigma)\sigma \in \mathbb{C}[S_n] \quad \text{(antisymm. the column transf.)}.$$

Young symmetrizer: $c_T := a_T \cdot b_T$.

Example:

i) $T = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}$ $c_T = a_T = \sum_{\sigma \in S_3} \sigma = 3!\,\pi_0$ (projector on trivial repr.),

ii) $T = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$ $c_T = b_T = \sum_{\sigma \in S_3} \text{sgn}(\sigma) \cdot \sigma = 3!\,\pi_1$ (projector on sign repr.),

iii) $T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$ $c_T = \underbrace{(e + (13))}_{a_T} \cdot (e - (12)) = e + (13) - (12) - (123) = 3\pi_2.$

④ With $A := \mathbb{C}[S_n]$,

$$V_T := \mathbb{C}[S_n] \cdot c_T = \{a \cdot c_T \mid a \in A\} \quad \text{is a subrepres. of } V_{\text{reg}}.$$

Examples

i)

$$T = \boxed{1}\boxed{2}\boxed{3}, \quad C(T) = \{e\}, \quad R(T) = \{e, (12), (23), (13), (123), (132)\} \equiv S_3.$$

$$\mathbb{C}[S_3] \cdot c_T = \mathbb{C}[S_3] \cdot \sum_{\sigma \in S_3} \sigma = \mathbb{C} \cdot c_T \quad \text{(1-dim)}$$

$$\left[\pi \sum_{\sigma \in S_3} \sigma = \sum_{\sigma \in S_3} (\pi\sigma) = \sum_{\sigma \in S_3} \sigma \right]$$

$$\rho_{\text{reg}}(\pi) c_T = \rho_{\text{reg}}(\pi) \sum_{\sigma \in S_3} \sigma = \sum_{\sigma \in S_3} (\pi\sigma) = \sum_{\sigma \in S_3} \sigma = c_T$$

$\rightarrow$ trivial repr.!

ii)

$$T = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}, \quad R(T) = \{e\}, \quad C(T) = S_3.$$

$$V_T = \mathbb{C}[S_3] c_T = \mathbb{C}[S_3] \sum_{\sigma \in S_3} \text{sign}(\sigma) \cdot \sigma = \mathbb{C} c_T,$$

$$\rho_{\text{reg}}(\pi) c_T = \pi \sum_{\sigma \in S_3} \text{sign}(\sigma)\sigma = \sum_{\sigma \in S_3} \text{sign}(\sigma) \cdot (\pi\sigma) = \text{sign}(\pi) \cdot \sum_{\sigma \in S_3} \text{sign}(\pi\sigma)(\pi\sigma)$$
$$= \text{sign}(\pi) \cdot \sum_{\sigma \in S_3} \text{sign}(\sigma) \cdot \sigma = \text{sign}(\pi) \cdot c_T$$

$\hookrightarrow \rho_{\text{reg}}|_{V_T}$ is the sign repres.

iii)

$$T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}, \quad c_T = e + (13) - (12) - (123).$$

$V_T = \mathbb{C}[S_3] \cdot c_T$ is 2-dimensional, spanned by c_T and $(12)c_T$, and $(13)c_T = c_T$. $\rho_{\text{reg}}|_{V_T}$ is the irreducible 2d repres.

Fact: (i) $\rho_{\text{reg}}|_{V_T}$ is an irreducible subrepresentation of ρ_{reg},
(ii) $V_T \cong V_{T'}$ iff $\lambda(T) = \lambda(T')$

$\rightarrow$ the irreducible repres. only depends on the underlying Young diagram,

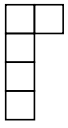
$\rightarrow$ there are as many copies of an irreducible representation corresponding to $(\lambda_1, ..., \lambda_n)$ as there are Young tableaux built on this Young diagram.

Recall:

$$\rho_{\text{reg}} = \rho_1^{\oplus \dim(\rho_1)} \oplus \cdots \oplus \rho_k^{\oplus \dim(\rho_k)}$$

$\rightarrow$ the irreducible represent. corresponding to a Young diagram has as dimension the number of ways you can make a Young tableau out of it.

Young diagram:

 $\mapsto$ irreducible repres. of S_n.

Young tableau:

$$\begin{array}{|c|c|}\hline 1 & 5 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \quad \mapsto \quad \text{subrepresentation of type } \ydiagram{2,1,1} \text{ in } \rho_{\text{reg}}.$$

- **Hook rule:** formula for the dimensions of irreducible representations of S_n.

  Labels irreducible repres.

  $\begin{array}{|c|c|c|}\hline 6 & 3 & \\ \hline 4 & 1 \\ \cline{1-2} 2 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}$

  Hook numbers: draw a hook through a box and count the number of boxes met by the hook, h_i.

  $$\boxed{\dim\left(\rho_{\ydiagram{3,2,1,1}}\right) = \frac{n!}{\prod h_i}} \quad \text{Hook rule}$$

Examples

i) $\underbrace{\begin{array}{|c|c|c|c|c|}\hline n & n-1 & n-2 & \dots & 1 \\ \hline\end{array}}_{n \text{ boxes}}$ S_n

$$\dim(\rho_{\ydiagram{4}}) = 1, \quad d = \frac{n!}{n(n-1)\cdots 1} = 1 \quad \text{(trivial repres.)}$$

ii) $\begin{array}{|c|}\hline n \\ \hline n-1 \\ \hline \vdots \\ \hline 1 \\ \hline\end{array}$

$$\dim\left(\rho_{\ydiagram{1,1,1,1}}\right) = 1, \quad d = \frac{n!}{n(n-1)\cdots} = 1 \quad \text{(sign repres.)}$$

iii) $n-1$ $\left\{\begin{array}{|c|c|}\hline n & 1 \\ \hline n-2 \\ \cline{1-1} \vdots \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}\right.$

$$d = \frac{n!}{n(n-2)\cdots 1} = (n-1)$$

Same for $\ydiagram{4,1}\cdots$.

- **Frobenius formula** (for characters of S_n)
  - Denote irreduc. repres. by partitions $(\lambda_1, ..., \lambda_n)$.
  - Denote conjugacy classes by the cycle structure $(k_1, ..., k_n)$.

  Define polynomials:

  $$P_{(k_1 \dots k_n)}(x_1 \dots x_n) = \prod_{i<j}(x_i - x_j)(x_1 + ... + x_n)^{k_1} \cdot (x_1^2 + ... + x_n^2)^{k_2} \cdot (x_1^3 + ... + x_n^3)^{k_3} \cdots (x_1^n + ... + x_n^n)^{k_n}.$$

  Then

  $$\chi_\lambda(C_k) = \left[P_{(k_1 \dots k_n)}(x_1 \dots x_n)\right]_{x_1^{l_1} \dots x_n^{l_n}}, \quad \lambda = (\lambda_1, ..., \lambda_n), \quad k = (k_1, ..., k_n),$$

  i.e. the coefficient of $x_1^{l_1} \dots x_n^{l_n}$ in P, with $l_i = \lambda_i + n - i$. Here λ_i are the row lengths of the Young diagram labelling the irreducible representation (a single row being the trivial representation).

Example: $G = S_3$, class $(\cdot\,\cdot)(\cdot)$:

$$P_{(1,1,0)}(x_1, x_2, x_3) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)(x_1 + x_2 + x_3)(x_1^2 + x_2^2 + x_3^2),$$

$\ydiagram{3}$ $\quad \lambda = (3,0,0) \quad l = (5,1,0) \quad \rightsquigarrow$ coeff. of $x_1^5 x_2$ in $P \quad \chi_\lambda(C_k) = 1$

 $\lambda = (1,1,1)$ $l = (3,2,1)$ $\rightsquigarrow$ coeff. of $x_1^3 x_2^2 x_3$ $\chi_\lambda(C_k) = -1$

 $\lambda = (2,1,0)$ $l = (4,2,0)$ $\rightsquigarrow$ coeff. of $x_1^4 x_2^2$ $\chi_\lambda(C_k) = 0$

$\rightsquigarrow$ C_2-column of the character table of $D_3 \cong S_3$.