6. Lie Groups / Lie Algebras

Lecture 7, Group Theory, Prof. Daniel Roggenkamp

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- Noether's theorem
- symmetries of space-time: Poincaré symmetries
- gauge symmetries: $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$

Def: A *Lie group* is a group which is also a smooth manifold, such that

$$\left.\begin{array}{l} m : G \times G \to G \text{ (multiplication)} \\ (\cdot)^{-1} : G \to G \quad \text{(inverse)} \end{array}\right\} \text{ are smooth maps.}$$

All examples we will consider are *matrix groups*, i.e. subgroups of $\mathrm{GL}(n)$, as such they are submanifolds of $\mathbb{R}^{n^2}/\mathbb{C}^{n^2}$. Often, they are described as level sets

$$\{f = 0\} \subset \mathrm{GL}(n)$$

for some smooth *regular* function $f : \mathbb{R}^n \to \mathbb{R}^k$ (regular: the derivative of f has maximum rank everywhere).

Examples

(i) $(\mathbb{R}, +)$ manifold:

$$\left.\begin{array}{l} m(x, y) = x + y \\ (x)^{-1} = -x \end{array}\right\} \text{ smooth maps}$$

(ii) $\mathrm{U}(1) = \{z \in \mathbb{C} \mid |z|^2 = 1\} = \{e^{i\varphi} \mid \varphi \in \mathbb{R}\} \cong S^1$ (unit circle):

$$\left.\begin{array}{l} m(z, w) = zw \\ (z)^{-1} = \overline{z} \end{array}\right\} \text{ smooth maps}$$

(iii) *(special) unitary group*

$$(\mathrm{S})\mathrm{U}(n) = \{A \in \mathrm{Mat}(n, n, \mathbb{C}) \mid A^\dagger A = \mathbb{1}, (\det A = +1)\}, \quad A^\dagger = \overline{A}^t$$

(iv) *(special) orthogonal group*

$$(\mathrm{S})\mathrm{O}(n) = \{A \in \mathrm{Mat}(n, n, \mathbb{R}) \mid A^t A = \mathbb{1}, (\det A = +1)\}$$

(v) *indefinite orthogonal groups*

$$\mathrm{O}(p, q) = \left\{ A \in \mathrm{Mat}(n, n, \mathbb{R}) \mid A^t \left(\begin{array}{c|c} \mathbb{1}_p & 0 \\ \hline 0 & -\mathbb{1}_q \end{array}\right) A = \left(\begin{array}{c|c} \mathbb{1}_p & 0 \\ \hline 0 & -\mathbb{1}_q \end{array}\right) \right\}, \quad n = p + q,$$

leaves invariant the signature of the (p, q) bilinear form, e.g. Lorentz group $\mathcal{L} = \mathrm{O}(1, 3)$.

(vi) *symplectic group*

$$\mathrm{Symp}(2n) = \left\{ A \in \mathrm{Mat}(2n, 2n, \mathbb{R}) \mid A^t \left(\begin{array}{c|c} 0 & \mathbb{1}_n \\ \hline -\mathbb{1}_n & 0 \end{array}\right) A = \left(\begin{array}{c|c} 0 & \mathbb{1}_n \\ \hline -\mathbb{1}_n & 0 \end{array}\right) \right\}$$

(vii) *Poincaré group*: symmetries of $\mathbb{R}^{1,3}$

$$P = \mathrm{O}(1, 3) \ltimes \mathbb{R}^{1,3},$$

semi-direct product: $(A, a)(B, b) = (AB, a + Ab)$.

Concepts in group theory carry over to Lie groups (but require smoothness), e.g.

A *homomorphism of Lie groups* is a smooth group homomorphism $\phi : G \to H$.

A *Lie subgroup* $H \subset G$ is a subgroup, s.t. the inclusion map is a smooth immersion.

Examples

(i) $(\mathbb{R}, +) \to \mathrm{U}(1)$, $x \mapsto e^{ix}$ hom. of Lie groups

(ii) $\mathrm{U}(1) \to \mathrm{U}(1)$, $z \mapsto z^n$, $n \in \mathbb{Z}$, hom. of Lie groups

(iii) $\det : \mathrm{GL}(n) \to (\mathbb{R}^\times, \cdot)$ is a hom. of Lie groups

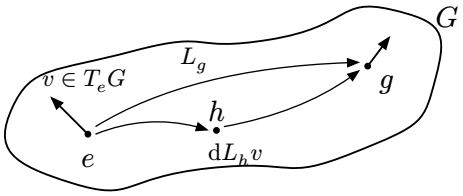
(iv) $\mathrm{O}(n) \subset \mathrm{O}(n+m)$, $A \mapsto \left(\begin{array}{c|c} A & 0 \\ \hline 0 & \mathbb{1}_m \end{array}\right)$ Lie subgroup

(v) $\mathrm{U}(1) \subset \mathrm{SU}(2)$, $z \mapsto \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}$ Lie subgroup

Combination of group theory with smooth manifolds $\hookrightarrow$ remarkable simplification by *linearization*.

Reason: All structures compatible with group multiplications are determined by their properties around $e \in G$. The left-multiplication map $L_g : G \to G$, $h \mapsto gh$ is a bijection of Lie groups.

$$\big(\mathrm{d}L_g\big)_e : T_eG \to T_gG$$



$\longrightarrow$ you get a vector field $X_v(g)$: *left-invariant* vector field

$$\boxed{\big(\mathrm{d}L_g\big)_h X_v(h) = X_v(gh)}$$

Much of the structure of G is captured by these vector fields.

$$\boxed{X_v(g) \coloneqq \big(\mathrm{d}L_g\big)_e v}$$

$\mathrm{d}L_g = \big(L_g\big)_*$ is the derivative of the map L_g, $L_e = \mathrm{id}_G$.

Aside about *vector fields*: In local coordinates

$$\boxed{X = \textstyle\sum_i X_i \partial_i} \quad \text{vector field on manifold } M,$$

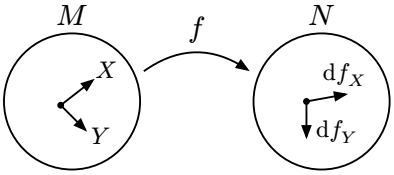
$$\begin{aligned} f : M \to \mathbb{R}, \quad X(f) &= \sum_i X_i(\partial_i f), \\ X(Yf) &= \sum_{i,j} X_i \partial_i\big(Y_j \partial_j f\big) = \sum_{i,j} X_i Y_j\big(\partial_i \partial_j f\big) + \big(X_i \partial_i Y_j\big)\partial_j f, \\ [X,Y](f) &= X(Yf) - Y(Xf) = \sum_{i,j}\big(X_i \partial_i Y_j - Y_i \partial_i X_j\big)\partial_j f = Z(f), \\ Z_j &= \sum_i \big(X_i \partial_i Y_j - Y_i \partial_i X_j\big), \end{aligned}$$

$$[\cdot,\cdot] : \mathrm{vf} \times \mathrm{vf} \to \mathrm{vf} \quad \text{Lie bracket of vector fields.}$$

Lie bracket: "calculates how a vector field changes along another one".

(i) $f : M \to N$

$$[\mathrm{d}fX, \mathrm{d}fY] = \mathrm{d}f([X,Y])$$



(ii) *Jacobi identity*

$$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$$

From (i) it follows that left-invariant vector fields close under the Lie bracket. Left-invariance: $\mathrm{d}L_g X_v = X_v$, so

$$\underbrace{\big[\mathrm{d}L_g X_v, \mathrm{d}L_g X_w\big]}_{=[X_v,X_w]} = \mathrm{d}L_g([X_v,X_w])$$

$\Longrightarrow$ the Lie bracket of left-inv. vector fields is left-inv.

$\longrightarrow T_eG$ is equipped with the structure of a Lie algebra:

$$v,w\in T_eG: \quad [v,w] \coloneqq X_{[v,w]} = [X_v,X_w].$$

Def: A *Lie algebra* is a vector space V together with a skew-symmetric, bilinear map

$$[\cdot,\cdot]: V\times V\to V$$

satisfying the Jacobi identity (skew-symmetric: $[v,w]=-[w,v]$).

Matrix Groups

$$G\subset \mathrm{GL}(n), \quad T_e\,\mathrm{GL}(n)=\mathrm{Mat}(n,n), \quad \mathrm{GL}(n)=\{A\in\mathrm{Mat}(n,n)\mid \det A\neq 0\}.$$

T_eG: smooth paths $h(t)\in G$, $h(0)=e$,

$$\mathrm{Mat}(n,n)\ni v=\dot h(0)\in T_eG,$$

$$\begin{aligned} L_g(h) &= gh \quad \text{(matrix multiplication)},\\ \mathrm{d}L_g v &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} L_g h(t) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} gh(t) = g\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} h(t) = g\cdot v,\\ X_v(g) &= g\cdot v. \end{aligned}$$

Lie bracket: $X=\sum_i X_i\partial_i$, $X(f)=\sum_i X_i(\partial_i f)$, $X(Yf)-Y(Xf)=[X,Y](f)$:

$$\begin{aligned} X_v f(g) &= \partial_g f(g)(g\cdot v),\\ X_vX_w f(g) &= \partial_g\big(\partial_g f(g)(g\cdot w)\big)g\cdot v = \partial_g^2 f(g)(g\cdot w, g\cdot v)+\partial_g f(g)gvw,\\ [X_v,X_w]f(g) &= \partial_g f(g)g(vw-wv)=\partial_g f(g)g([v,w]),\\ [X_v,X_w](g) &= g[v,w]=X_{[v,w]} \quad \text{(matrix commutator)}. \end{aligned}$$

$T_eG\subset\mathrm{Mat}(n,n)$ has the structure of a Lie algebra, where

$$[v,w]=vw-wv \quad \text{(matrix multiplication)}$$

(the matrix commutator satisfies the Jacobi identity).

Example: $G=\mathrm{O}(n)=\{A\in\mathrm{Mat}(n,n,\mathbb{R})\mid A^tA=\mathbb{1}_n\}$ is the level set

$$G=\{f(A)=0\}, \quad f(A)=A^tA-\mathbb{1}_n, \quad T_eG=\ker(\mathrm{d}f_e).$$

Take a path $A(t)$ s.t. $A(0)=\mathbb{1}_n$, $\dot A(0)=a$:

$$\mathrm{d}f_e a = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(A(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(A^tA-\mathbb{1}_n) = a^t+a$$

$$T_eG = \{a \in \mathrm{Mat}(n,n) \mid a^t = -a\}, \quad \dim(G) = \frac{n}{2}(n-1). \qquad \text{shape: } \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{pmatrix}$$

Basis of T_eG: $E_{(i,j)}$, $i<j$, the matrix with entry $+1$ in row i, column j, entry -1 in row j, column i and zeros elsewhere,

$$E_{(i,j)} = \begin{pmatrix} 0 & & \cdots & \\ & \ddots & 1 & \\ & & \ddots & \\ \vdots & -1 & \ddots & \\ & & & 0 \end{pmatrix} \leftarrow \text{row } i \text{ holds the } 1 \text{ (column } j\text{), row } j \text{ the } -1 \text{ (column } i\text{)},$$

$$\left[E_{(i,j)}, E_{(r,s)}\right] = \delta_{jr}E_{(i,s)} + \delta_{is}E_{(j,r)} - \delta_{ir}E_{(j,s)} - \delta_{js}E_{(i,r)}.$$

$n=3$: with the antisymmetric tensor ε_{abc},

$$E_a = -\frac{1}{2}\varepsilon_{abc}E_{(b,c)}, \quad [E_a, E_b] = \varepsilon_{abc}E_c.$$

Often, Lie algebras are given in terms of a basis.

Def: A *homomorphism of Lie algebras* is a linear map $\varphi : V \to W$, s.t.

$$[\varphi(v), \varphi(w)] = \varphi([v,w]).$$

Def: A *sub-Lie algebra* $V' \subset V$ is a subspace s.t.

$$[v,w] \in V' \quad \forall v, w \in V'.$$

Information about Lie groups is contained in their Lie algebras.

	Lie group		**Lie algebra**
(i)	G	$\mapsto$	$T_eG = \mathfrak{g}$
(ii)	$\phi : G \to H$ hom. of Lie groups	$\mapsto$	$\varphi := \mathrm{d}\phi_e : \underbrace{T_eG}_{\mathfrak{g}} \to \underbrace{T_eH}_{\mathfrak{h}}$ hom. of Lie algebras
(iii)	$H \subset G$ subgroup	$\mapsto$	$T_eH = \mathfrak{h} \subset \mathfrak{g} = T_eG$ subalgebra
(iv)	unique simply connected Lie group $\tilde{G}$ with $\mathrm{Lie}\left(\tilde{G}\right) = \mathfrak{g}$ *	$\leftarrow\!\shortmid$	$\mathfrak{g}$
(v)	$\exists!$ connected $H \subset G$	$\leftarrow\!\shortmid$	$\mathfrak{h} \subset \mathfrak{g}$

* but any $G = \tilde{G}/\Gamma$, Γ a discrete subgroup of $Z\left(\tilde{G}\right)$, also has the same Lie algebra.

Proof of (ii):

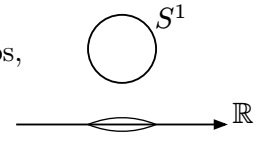
$$\begin{aligned} \phi(gh) &= \phi(g)\phi(h), \\ \phi \circ L_g &= L_{\phi(g)} \circ \phi, \\ \mathrm{d}\phi \circ \mathrm{d}L_g &= \mathrm{d}L_{\phi(g)} \circ \mathrm{d}\phi, \\ \mathrm{d}\phi X_v(g) &= X_{\mathrm{d}\phi_e v}(\phi(g)), \end{aligned}$$

$$\begin{aligned} X_{[\mathrm{d}\phi_e v,\, \mathrm{d}\phi_e w]} &= \left[X_{\mathrm{d}\phi_e v}, X_{\mathrm{d}\phi_e w}\right] = [\mathrm{d}\phi X_v(g), \mathrm{d}\phi X_w(g)] \\ &= \mathrm{d}\phi[X_v, X_w](g) = \mathrm{d}\phi X_{[v,w]} = X_{\mathrm{d}\phi_e[v,w]}. \end{aligned}$$

From this follows (ii).

(iii) This follows from looking at the inclusion.

Def: *Simply connected* means connected and there are no non-contractible loops, e.g. S^1 is *not* simply connected.



Continuing the table from above:

	Lie group		**Lie algebra**
(vi)	$\exists!\phi : G \to H$, $\mathrm{d}\phi_e = \varphi$ G simply connected, H connected	$\leftarrow\!\shortmid$	$\varphi : \mathfrak{g} \to \mathfrak{h}$
(vii)	$G \cong H$ if both are simply connected	$\leftarrow\!\shortmid$	$\mathfrak{g} \cong \mathfrak{h}$

In particular:

repr. of $G \mapsto$ repr. of $\mathfrak{g}$,
repr. of $G \leftarrow\!\shortmid$ repr. of $\mathfrak{g}$ but only if G is simply connected.

If G is not simply connected, then not all repr. of $\mathfrak{g}$ integrate to repr. of G. E.g. SO(3):

$$\mathfrak{so}(3) = \mathrm{Lie}(\mathrm{SO}(3)) \cong \mathfrak{su}(2) = \mathrm{Lie}(\mathrm{SU}(2)).$$

Repr. of $\mathfrak{su}(2)$ are denoted by spin $j \in \frac{1}{2}\mathbb{N}$, but only the integral spins give rise to repr. of SO(3).