Exponential Map and the Lorentz Group

Lecture 8, Group Theory, Prof. Daniel Roggenkamp

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Summary of last week's lecture

	Lie group		**Lie algebra**
(i)	G	$\mapsto$	$\mathfrak{g} \cong T_eG$
(ii)	$\phi : G \to H$	$\mapsto$	$\varphi = \mathrm{d}\phi_e : \mathfrak{g} \to \mathfrak{h}$
(iii)	$H \subset G$	$\mapsto$	$\mathfrak{h} \subset \mathfrak{g}$
(iv)	$\exists!$ simply connected Lie group $\tilde{G}$ other: $G = \tilde{G}/\Gamma$, $\Gamma \subset Z\big(\tilde{G}\big)$	$\hookleftarrow$	$\mathfrak{g}$
(v)	unique connected $H \subset G$	$\hookleftarrow$	$\mathfrak{h} \subset \mathfrak{g}$
(vi)	$\exists!\phi : G \to H$, G simply connected, H connected	$\hookleftarrow$	$\varphi : \mathfrak{g} \to \mathfrak{h}$

Representations

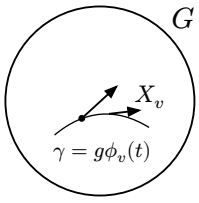
$$\begin{aligned} \phi : G \to \mathrm{GL}(V) \quad &\mapsto \quad \varphi = (\mathrm{d}\phi)_e : \mathfrak{g} \to \mathfrak{gl}(V), \\ \exists!\phi : \tilde{G} \to \mathrm{GL}(V) \quad &\hookleftarrow \quad \varphi : \mathfrak{g} \to \mathfrak{gl}(V), \end{aligned}$$

where $\tilde{G}$ is the simply connected Lie group with Lie algebra $\mathfrak{g}$. For $G = \tilde{G}/\Gamma$: $\ker(\phi) \supset \Gamma$.

Exponential map

G: Lie group, $\mathrm{Lie}(G) = \mathfrak{g}$. For every $v \in \mathfrak{g}$ we can define the subalgebra $\mathfrak{h} = \mathrm{span}(v) \subset \mathfrak{g}$. Using (v) from the above table $\longrightarrow$ 1-parameter subgroup $H \subset G$ (unique connected):

$$\begin{gathered} \phi_v : (\mathbb{R}, +) \to G \quad | \quad \varphi_v : \mathfrak{h} \to \mathfrak{g}, \quad v \mapsto v, \\ \phi_v(s)\phi_v(t) = \phi_v(s+t), \\ \big(\mathrm{d}\phi_v\big)_e = v. \end{gathered}$$



(i) $\phi_v(t) : \mathbb{R} \to G$ (integral curve of X_v),

$$\dot{\gamma}(t) = X_v(\gamma(t)),$$

$\gamma = g \cdot \phi_v(t)$ integral curve of X_v through $g \in G$:

$$\begin{aligned} \dot{\gamma}(t) &= \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \gamma(t+s) = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} g\phi_v(t)\phi_v(s) \\ &= \mathrm{d}\Big(L_{g\phi_v(t)}\Big)_e \underbrace{\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \phi_v(s)}_{v} = \mathrm{d}\Big(L_{g\phi_v(t)}\Big)_e v = X_v(\gamma(t)). \end{aligned}$$

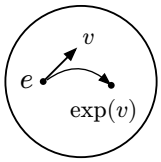
(ii) $\phi_v(t) = \phi_{vt}(1)$ (both integral curves of the same vector field).

Define: $\exp : \mathfrak{g} \to G$, $v \mapsto \phi_v(1)$.

(i) $\exp(tv)\exp(sv) = \exp((s+t)v)$

(ii) $\underbrace{\mathrm{d}(\exp)_0}_{\mathrm{id}_{\mathfrak{g}}} : \mathfrak{g} \to \mathfrak{g}$, $v \mapsto v$

(iii) $\phi : G \to H \Longrightarrow \phi(\exp(v)) = \exp(\mathrm{d}\phi_e v)$



Matrix groups

$$\exp = \mathrm{Exp} : \mathrm{Mat}(n,n) \to \mathrm{GL}(n), \quad v \mapsto \sum_{i \geq 0} \frac{v^i}{i!} \quad \text{(matrix multiplication)}$$

Baker–Campbell–Hausdorff formula

$$\exp(v)\exp(w) = \exp\Big(v + w + \frac{1}{2}[v,w] + \frac{1}{12}([v,[v,w]] + [w,[w,v]]) + \cdots\Big)$$

The Lorentz group

Minkowski space: $(\mathbb{R}^{1,3}, \eta = \mathrm{diag}(1,-1,-1,-1))$,

$$\mathcal{L} = \mathrm{O}(1,3) = \{\Lambda \in \mathrm{Mat}(4,4) \mid \Lambda^t \eta \Lambda = \eta\},$$

$$\Lambda = \left(\begin{array}{c|c} \lambda_{00} & \lambda_2^t \\ \hline \lambda_1 & \hat{\Lambda} \end{array}\right), \quad \lambda_{00} \in \mathbb{R}; \quad \lambda_1, \lambda_2 \in \mathbb{R}^3; \quad \hat{\Lambda} \in \mathrm{Mat}(3,3),$$

$$\Lambda^t \eta \Lambda = \left(\begin{array}{c|c} \lambda_{00}^2 - \lambda_1^t \lambda_1 & \lambda_{00}\lambda_2^t - \lambda_1^t \hat{\Lambda} \\ \hline \lambda_2 \lambda_{00} - \hat{\Lambda}^t \lambda_1 & -\hat{\Lambda}^t \hat{\Lambda} + \lambda_2 \lambda_2^t \end{array}\right) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -\mathbb{1}_3 \end{array}\right). \qquad (*)$$

Polar decomposition of $\mathrm{O}(1,3)$

Note that $\eta^t = \eta$, $\eta^2 = \mathbb{1}_4 \Longrightarrow \eta \in \mathrm{O}(1,3)$, $\Lambda^t = \eta \Lambda^{-1} \eta \in \mathrm{O}(1,3)$

$$\Longrightarrow \Lambda^t \Lambda \in \mathrm{O}(1,3), \text{symmetric}, > 0,$$

$$\mathcal{L} := \sqrt{\Lambda^t \Lambda}, \quad \mathcal{L}^t = \mathcal{L}, \quad \mathcal{L} > 0, \quad \mathcal{L} \in \mathrm{O}(1,3),$$

$$R := \Lambda \mathcal{L}^{-1} \in \mathrm{O}(1,3),$$

$$R^t R = \left(\mathcal{L}^{-1}\right)^t \Lambda^t \Lambda \mathcal{L}^{-1} = \mathbb{1}_4 \Longrightarrow R \in \mathrm{O}(1,3) \cap \mathrm{O}(4)$$

$$\Longrightarrow R \in \mathrm{O}(1) \times \mathrm{O}(3), \quad R = \left(\begin{array}{c|ccc} \varepsilon & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & Q & \\ 0 & & & \end{array}\right), \quad \varepsilon \in \mathrm{O}(1) = \{\pm 1\}, \quad Q \in \mathrm{O}(3),$$

$$\Lambda = \left(\begin{array}{c|c} \varepsilon & 0 \\ \hline 0 & Q \end{array}\right)\mathcal{L}, \quad \mathcal{L} = \left(\begin{array}{c|c} \lambda_{00} & \lambda_2^t \\ \hline \lambda_1 & \hat{\Lambda} \end{array}\right), \quad \hat{\Lambda}^t = \hat{\Lambda}.$$

$$(*) \Longrightarrow \lambda_{00} = \sqrt{1 + \lambda^t \lambda}, \quad \hat{\Lambda} = \sqrt{\mathbb{1} + \lambda \lambda^t},$$

$$\Lambda = \underbrace{\left(\begin{array}{c|c} \varepsilon & 0 \\ \hline 0 & Q \end{array}\right)}_{\substack{\varepsilon\text{: time reversal,} \\ Q\text{: space transf.}}} \cdot \underbrace{\left(\begin{array}{c|c} \sqrt{1+\lambda^t\lambda} & \lambda^t \\ \hline \lambda & \sqrt{\mathbb{1} + \lambda\lambda^t} \end{array}\right)}_{\text{boosts}},$$

$$\mathrm{O}(1,3) \overset{\text{homeo}}{\cong} \mathrm{O}(1) \times \mathrm{O}(3) \times \mathbb{R}^3$$

$\longrightarrow$ 4 components distinguished by

$$\begin{aligned} \det(\Lambda) = \varepsilon \det(Q) = \pm 1 \quad & (+1 : \text{proper, preserve orientation}), \\ \operatorname{sign}(\lambda_{00}) = \varepsilon = \pm 1 \quad & (+1 : \text{orthochronous, time-direction preserved}). \end{aligned}$$

The connected component of $e \in \mathrm{O}(1,3)$ is called $\mathrm{O}(1,3)_0 \lhd \mathrm{O}(1,3)$,

$\mathrm{O}(1,3)/\mathrm{O}(1,3)_0 = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{\mathbb{1}, P, T, PT\}$, $P = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -\mathbb{1}_3 \end{array}\right)$ parity reversal, $T = \left(\begin{array}{c|c} -1 & 0 \\ \hline 0 & \mathbb{1}_3 \end{array}\right)$ time reversal, $PT = -\mathbb{1}_4$.

Lie algebra $\mathfrak{o}(1,3)$

$\mathrm{O}(1,3)$ is a level set: $f(\Lambda) = \Lambda^t \eta \Lambda - \eta$,

$$\begin{aligned} \mathrm{d}f_e v &= v^t \eta + \eta v, \\ T_e\ \mathrm{O}(1,3) = \mathfrak{o}(1,3) &= \ker(\mathrm{d}f_e) = \{v \mid v^t = -\eta v \eta\}, \\ v &= \left(\begin{array}{c|c} v_0 & c^t \\ \hline d & \hat{v} \end{array}\right) = \left(\begin{array}{c|c} -v_0 & +d^t \\ \hline +c & -\hat{v}^t \end{array}\right) \\ \Longrightarrow v_0 = 0, \quad c = d, \quad & \hat{v}^t = -\hat{v}, \quad v = \left(\begin{array}{c|c} 0 & c^t \\ \hline c & \hat{v} \end{array}\right). \end{aligned}$$

Basis:

$$\omega_1 = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right), \quad \omega_2 = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right), \quad \omega_3 = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

$$K_1 = \left(\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \quad K_2 = \left(\begin{array}{c|ccc} 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \quad K_3 = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right),$$

$\mathfrak{o}(1,3) = \operatorname{span}\{\omega_i, K_i\}$ $\longleftarrow$ 6-dimensional real Lie algebra,

$$\boxed{\begin{aligned} [\omega_i, \omega_j] &= \sum_k \varepsilon_{ijk} \omega_k, \\ [K_i, K_j] &= -\sum_k \varepsilon_{ijk} \omega_k, \\ [\omega_i, K_j] &= \sum_k \varepsilon_{ijk} K_k \end{aligned}}$$

$$\mathrm{SL}_2(\mathbb{C}) = \{A \in \mathrm{Mat}(2,2,\mathbb{C}) \mid \det(A) = 1\},$$

$$\begin{aligned} \mathfrak{sl}_2(\mathbb{C}) &= \{a \in \mathrm{Mat}(2,2,\mathbb{C}) \mid \operatorname{tr}(a) = 0\} \\ &= \operatorname{span}_{\mathbb{C}} \left\{ j_1 = -\frac{i}{2}\sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, j_2 = -\frac{i}{2}\sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, j_3 = -\frac{i}{2}\sigma_3 = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \right\}, \end{aligned}$$

where σ_i are the Pauli matrices, and

$$[j_a, j_b] = \sum_c \varepsilon_{abc} j_c$$

$\longrightarrow$ complex 3-dimensional Lie algebra.

Choose a real structure:

$$\mathfrak{sl}_2(\mathbb{C}) = \operatorname{span}_{\mathbb{R}}\{j_1, j_2, j_3\} \otimes_{\mathbb{R}} \mathbb{C},$$

$$\varphi : (\mathfrak{sl}_2(\mathbb{C}))_{\mathbb{R}} \to \mathfrak{o}(1,3), \quad j_a \mapsto \omega_a, \quad ij_a \mapsto K_a$$

is an isomorphism of real Lie algebras. $\mathrm{SL}(2,\mathbb{C})$ is in fact simply connected!

$\Longrightarrow \varphi$ integrates to a unique

$$\phi : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{O}(1,3)_0, \quad \mathrm{d}\phi_e = \varphi,$$

which is possibly a covering.

Identify

$$\mathbb{R}^{1,3} \xrightarrow{\cong} i\mathfrak{u}(2) \text{ (hermitian } 2\times 2 \text{ matrices)}, \quad x \mapsto \sigma(x) = \sum_i x_i \sigma_i = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix},$$

$$x^t \eta x = \det(\sigma(x)).$$

Now $\mathrm{SL}(2,\mathbb{C})$ acts on $i\mathfrak{u}(2)$:

$$\sigma(x) \mapsto A\sigma(x)A^\dagger, \quad A \in \mathrm{SL}(2,\mathbb{C}).$$

But this action preserves η:

$$\det\big(A\sigma(x)A^\dagger\big) = \det(\sigma(x)) \quad \text{since } \det(A) = 1$$

$$\Longrightarrow A\sigma(x)A^\dagger = \sigma(\Psi(A)\cdot x), \quad \text{where } \Psi : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{O}(1,3).$$

Check: $\mathrm{d}\Psi_e = \varphi \Longrightarrow \Psi = \phi$.

$A(t)$ path in $\mathrm{SL}(2,\mathbb{C})$: $A(0) = \mathbb{1}$, $\dot{A}(0) = a$,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} A(t)\sigma(x)A^\dagger(t) = a\sigma(x) + \sigma(x)a^\dagger = \sigma((\mathrm{d}\Psi_e a)x),$$

$$a = j_i = -\frac{i}{2}\sigma_i \Longrightarrow \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} A(t)\sigma(x)A^\dagger(t) = -\frac{i}{2}[\sigma_i, \sigma(x)] = \sum \sigma_c(\omega_i x)_c = \sigma(\omega_i x),$$

$$a = ij_i = \frac{1}{2}\sigma_i \Longrightarrow \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} A(t)\sigma(x)A^\dagger(t) = \sigma(K_i x),$$

$$\Longrightarrow \varphi = \mathrm{d}\Psi_e : j_i \mapsto \omega_i, \quad ij_i \mapsto K_i$$
$$\Longrightarrow \Psi = \phi,$$
$$\ker(\phi) = \big\{A \in \mathrm{SL}(2,\mathbb{C}) \mid A\sigma A^\dagger = \sigma \;\forall \sigma\big\} = \{\pm \mathbb{1}_2\}.$$

Remark: complexification

$$\mathfrak{o}(1,3)_{\mathbb{C}} = \mathfrak{o}(1,3) \otimes_{\mathbb{R}} \mathbb{C} \quad \longrightarrow \quad \text{6-dimensional, complex Lie algebra,}$$

basis: $J_i^{\pm} := \frac{1}{2}(\omega_i \pm iK_i)$, a complex basis of $\mathfrak{o}(1,3)_{\mathbb{C}}$:

$$\big[J_i^+, J_j^-\big] = 0 \quad \forall i,j, \quad \big[J_i^{\pm}, J_j^{\pm}\big] = \sum_k \varepsilon_{ijk} J_k^{\pm},$$

$$\mathfrak{o}(1,3)_{\mathbb{C}} \xrightarrow{\cong} \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}), \quad J_i^+ \mapsto (j_i, 0), \quad J_i^- \mapsto (0, j_i),$$

an isomorphism of complex Lie algebras:

$$\boxed{\mathfrak{o}(1,3)_{\mathbb{C}} \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})}$$