Quantum Field Theory I

Lecture Revisions

Janosh Riebesell, winter semester 2014/15

Revision 1 (13.10.2014)

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Literature:

- Lecture notes "Quantum Field Theory I & II" by Timo Weigand, Chapters 1 to 6
- "An Introduction to Quantum Field Theory" by Peskin and Schröder

Tutorials: organized by Dr. Viraf Mehta (v.mehta@thphys.uni-heidelberg.de), online registration
The course is passed by successfully standing the written examination at the end of the semester.
Plan:

1. The free scalar field
2. The interacting scalar field
3. Quantizing spin-$\frac{1}{2}$-fields
4. Quantizing spin-1-fields
5. Quantum electrodynamics (QED)
6. Classical non-abelian gauge theory

Revision 2 (17.10.2014)

Why Quantum Field Theory?

We will focus on elementary particle physics, hence relativistic QFT. However, virtually the same methods play a role in nuclear, atomic and condensed matter physics. In any relativistic field theory, the particle number is not conserved; since $E^2 = c^2\boldsymbol{p}^2 + m^2c^4$, energy can always be converted into particles and vice versa. This requires a *multiparticle* framework different from quantum mechanics.

QFT is a change of perspective from quantum mechanics in the following regards:

- The fundamental entities are not the particles but rather "the field" – an abstract object that "penetrates" spacetime.
- Particles are the "excitations" of the field.

Revision 3 (20.10.2014)

- Free scalar field theory ($\phi^* = \phi$): $S = \frac{1}{2}\int_{\mathbb{R}^4} \mathrm{d}^4x \left[\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right]$
- Euler-Lagrange: $\dfrac{\partial\mathcal{L}}{\partial\phi} = \partial_\mu \dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}$
- Symmetry: $\phi \to \phi + \epsilon\cdot\delta\phi + \mathcal{O}(\epsilon^2)$ $\qquad (\partial\phi = \partial_\nu\phi(x^\mu))$
  $\mathcal{L} \to \mathcal{L} + \epsilon\partial_\mu F^\mu + \mathcal{O}(\epsilon^2)$
- Noether current: $j^\mu = \dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi - F^\mu$
- Conserved charge: $\partial_\mu j^\mu = 0$, $Q = \int_{\mathbb{R}^3} \mathrm{d}^3x\, j^0$

Revision 4 (22.10.2014)

- Energy-momentum tensor: $T^{\mu\nu} = \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - \eta^{\mu\nu}\mathcal{L}$

  For this tensor, one has $\partial_\mu T^{\mu\nu} = 0$, $E = \int_{\mathbb{R}^3} \mathrm{d}^3x\, T^{00}$, and $p^i = \int_{\mathbb{R}^3} \mathrm{d}^3x\, T^{0i}$.

  Free scalar field: $E = \displaystyle\int_{\mathbb{R}^3} \mathrm{d}^3x \left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right]$, $\boldsymbol{p} = -\int_{\mathbb{R}^3} \mathrm{d}^3x \left(\dot{\phi}\nabla\phi\right)$
- Canonical quantization
  i) Conjugate momentum density: $\pi(\boldsymbol{x}, t) = \dfrac{\partial \mathcal{L}}{\partial\left(\dot{\phi}(\boldsymbol{x}, t)\right)}$
  ii) $H = \displaystyle\int_{\mathbb{R}^3} \mathrm{d}^3x\, \mathcal{H} = \int_{\mathbb{R}^3} \mathrm{d}^3x \left[\pi\dot{\phi} - \mathcal{L}\right] \overset{\text{free theory}}{=} \int_{\mathbb{R}^3} \mathrm{d}^3x \left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right]$
  iii) Promote $\phi(\boldsymbol{x})$ and $\pi(\boldsymbol{x})$ to operators:
  - $[\phi(\boldsymbol{x}), \pi(\boldsymbol{y})] = i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y})$
  - $[\phi(\boldsymbol{x}), \phi(\boldsymbol{y})] = [\pi(\boldsymbol{x}), \pi(\boldsymbol{y})] = 0$

Revision 5 (27.10.2014)

- Mode expansion:

$$\phi(\boldsymbol{x}) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}} \left(a(\boldsymbol{p})e^{i\boldsymbol{p}\boldsymbol{x}} + a^\dagger(\boldsymbol{p})e^{-i\boldsymbol{p}\boldsymbol{x}}\right)$$

$$\pi(\boldsymbol{x}) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3p}{(2\pi)^3} (-i)\sqrt{\frac{\omega_{\boldsymbol{p}}}{2}} \left(a(\boldsymbol{p})e^{i\boldsymbol{p}\boldsymbol{x}} - a^\dagger(\boldsymbol{p})e^{-i\boldsymbol{p}\boldsymbol{x}}\right)$$

- $\left[a(\boldsymbol{p}), a^\dagger(\boldsymbol{q})\right] = (2\pi)^3\delta^{(3)}(\boldsymbol{p} - \boldsymbol{q})$
  $[a(\boldsymbol{p}), a(\boldsymbol{q})] = \left[a^\dagger(\boldsymbol{p}), a^\dagger(\boldsymbol{q})\right] = 0 \quad \forall \boldsymbol{p}, \boldsymbol{q}$
- 4-momentum: $P^\mu = \displaystyle\int_{\mathbb{R}^3} \frac{\mathrm{d}^3p}{(2\pi)^3} p^\mu a^\dagger(\boldsymbol{p})a(\boldsymbol{p})$, where $p^\mu = (p^0, \boldsymbol{p})$ and $p^0 = \omega_{\boldsymbol{p}} = E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$
  $[P^\mu, a(\boldsymbol{p})] = -p^\mu a(\boldsymbol{p})$
  $\left[P^\mu, a^\dagger(\boldsymbol{p})\right] = p^\mu a^\dagger(\boldsymbol{p})$
- Vacuum: $a(\boldsymbol{p})|0\rangle = 0 \quad \forall \boldsymbol{p}$
  $a^\dagger(\boldsymbol{p})$ creates 1-particle state with momentum $\boldsymbol{p}$: $P^\mu a^\dagger(\boldsymbol{p})|0\rangle = p^\mu a^\dagger(\boldsymbol{p})|0\rangle$
  N-particle state $a^\dagger(\boldsymbol{p}_1)...a^\dagger(\boldsymbol{p}_N)|0\rangle$ with energy $E = \sum_{i=1}^N E_{\boldsymbol{p},i}$ and momentum $\boldsymbol{p} = \sum_{i=1}^N \boldsymbol{p}_i$.
  $\Rightarrow$ QFT is a multi-particle framework

Revision 6 (29.10.2014)

- N-particle momentum eigenstates
  $|\boldsymbol{p}_1, ..., \boldsymbol{p}_N\rangle = \prod_{i=1}^N \sqrt{2\omega_{\boldsymbol{p}_i}} a^\dagger(\boldsymbol{p}_i)|0\rangle$
  $\langle \boldsymbol{q}|\boldsymbol{p}\rangle = (2\pi)^3 2\omega_{\boldsymbol{p}} \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q})$
- Position eigenstates
  $|\boldsymbol{x}\rangle = \phi(\boldsymbol{x})|0\rangle = \displaystyle\int_{\mathbb{R}^3} \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{2\omega_{\boldsymbol{p}}} e^{-i\boldsymbol{p}\boldsymbol{x}}|\boldsymbol{p}\rangle$[1]

[1] All these mode expansions and creating a particle at position $\boldsymbol{x}$ as simple as $|\boldsymbol{x}\rangle = \phi(\boldsymbol{x})|0\rangle$ are only valid in the free theory without interaction and therefore without any influence of particle creation on nearby particles.

- Renormalization $H = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} \omega_{\boldsymbol{p}} a^\dagger(\boldsymbol{p}) a(\boldsymbol{p}) + \Delta_H$,

  where $\epsilon_0 = \frac{\Delta_H}{\mathrm{Vol}_{\mathbb{R}^3}} = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{\omega_{\boldsymbol{p}}}{2} \to \infty$ (UV divergent)

  To renormalize, we absorb ϵ_0 into V_0 in the Lagrangian $\mathcal{L} = \frac{1}{2} \left(\partial_\mu \phi\right)^2 - \frac{1}{2} m^2 \phi^2 - V_0$

Revision 7 (3.11.2014)

- Complex scalar field
  $\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi$
  $\pi(t, \boldsymbol{x}) = \dot{\phi}^\dagger(t, \boldsymbol{x})$, $\pi^\dagger(t, \boldsymbol{x}) = \dot{\phi}(t, \boldsymbol{x})$
  $[\phi(\boldsymbol{x}), \pi(\boldsymbol{y})] = \left[\phi^\dagger(\boldsymbol{x}), \pi^\dagger(\boldsymbol{y})\right] = i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y})$, all other commutators vanish
- Mode expansion
  $\phi(\boldsymbol{x}) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}} \left(a(\boldsymbol{p}) e^{i\boldsymbol{p}\boldsymbol{x}} + b^\dagger(\boldsymbol{p}) e^{-i\boldsymbol{p}\boldsymbol{x}}\right)$
  $a^\dagger(\boldsymbol{p})|0\rangle$: particle with momentum $\boldsymbol{p}$ and charge -1,
  $b^\dagger(\boldsymbol{p})|0\rangle$: particle with momentum $\boldsymbol{p}$ and charge 1
- Heisenberg Picture
  $\phi(t, \boldsymbol{x}) = \phi^{(H)}(t, \boldsymbol{x}) = e^{iH^{(S)}t} \phi^{(S)}(\boldsymbol{x}) e^{-iH^{(S)}t}$
  equal-time commutation relations $[\phi(t, \boldsymbol{x}), \pi(t, \boldsymbol{y})] = i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y})$
  Klein-Gordon operator equation $(\partial^2 + m^2)\phi(t, \boldsymbol{x}) = 0$

Revision 8 (5.11.2014)

- Propagator: 2-point correlation function
  $D(x - y) = \langle 0|\phi(x)\phi(y)|0\rangle \overset{\text{free scalar theory}}{=} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} e^{-ip(x-y)}$
- Commutator
  $\Delta(x - y) = [\phi(x), \phi(y)] = D(x - y) - D(y - x)$

Revision 9 (10.11.2014)

- Feynman propagator
  $D_{\mathrm{F}}(x - y) = \langle 0|T\phi(x)\phi(y)|0\rangle = \begin{cases} \langle 0|\phi(x)\phi(y)|0\rangle & x^0 \geq y^0 \\ \langle 0|\phi(y)\phi(x)|0\rangle & x^0 < y^0 \end{cases}$
  - Free theory (without interaction) $D_{\mathrm{F}}(x - y) = \int_{\mathbb{R}^4} \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}$

    where the $i\epsilon$-term represents time-ordering

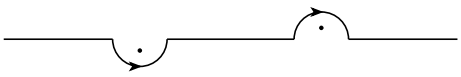


Figure 1: This integration path in the complex momentum plane yields the (time-ordered) Feynman propagator $D_{\mathrm{F}}(x - y)$.

- causally retarded and advanced propagators require different integration schemes
  - $D_{\mathrm{A}}(x - y)$



Figure 2: This integration path in the complex momentum plane yields the advanced propagator $D_{\mathrm{A}}(x - y)$.

  - $D_{\mathrm{R}}(x - y)$

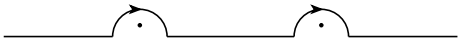


Figure 3: This integration path in the complex momentum plane yields the retarded propagator $D_{\mathrm{R}}(x-y)$.

- $D_{\mathrm{R}}(x-y)/D_{\mathrm{A}}(x-y)$ propagates information forward (backward) in time
- $D_{\mathrm{F}}(x-y)$ propagates positive frequency modes (e^{-ipx}) forward in time and negative frequency modes (e^{ipx}) backward in time

Revision 10 (12.11.2014)

Interacting scalar field theory

- $V(\phi) = \frac{1}{2}m_0^2\phi^2 + \frac{1}{3!}g\phi^3 + \frac{1}{4!}\lambda\phi^4 + ...$
- $H|\lambda_{\boldsymbol{p}}\rangle = E_{\boldsymbol{p}}(\lambda)|\lambda_{\boldsymbol{p}}\rangle$
  $\boldsymbol{P}|\lambda_{\boldsymbol{p}}\rangle = \boldsymbol{p}|\lambda_{\boldsymbol{p}}\rangle$
  - The state parameter $\lambda_{\boldsymbol{p}}$ denotes 1-particle, multi-particle and bound states; it hides much of the complexity that comes with interaction.
- $\mathbb{1} = |\Omega\rangle\langle\Omega| + \sum_\lambda \int_{\mathbb{R}^3} \frac{\mathrm{d}^3p}{(2\pi)^3}\,\frac{1}{2E_{\boldsymbol{p}}(\lambda)}\,|\lambda_{\boldsymbol{p}}\rangle\langle\lambda_{\boldsymbol{p}}|$
- $\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \sum_\lambda \int_{\mathbb{R}^3} \frac{\mathrm{d}^3p}{(2\pi)^3}\,\frac{1}{2E_{\boldsymbol{p}}(\lambda)}\,e^{-ip(x-y)}\underbrace{|\langle\Omega|\phi(0)|\lambda_0\rangle|^2}_{\text{field strength}}$

  Note: $E_{\boldsymbol{p}}(\lambda) = \sqrt{\boldsymbol{p}^2 + m_\lambda^2}$ involves the rest energy m_λ of the state λ (e.g. the invariant mass of a multi-particle or bound state) rather than a single particle mass m

Revision 11 (17.11.2014)

- $\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \int_0^\infty \frac{\mathrm{d}M^2}{2\pi}\,\rho(M^2)D_{\mathrm{F}}(x-y,M^2)$

  spectral density $\rho(M^2) = \sum_\lambda 2\pi\delta(M^2-m_\lambda^2)\;|\langle\Omega|\phi(0)|\lambda_0\rangle|^2$
- Lesson: the full propagator $\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle$ yields the mass "m" as the first pole
- The propagator is a two-point correlation function.
- n-point correlation function: $\langle\Omega|T\phi(x_1)\phi(x_2)...\phi(x_n)|\Omega\rangle$
- scattering $|i\rangle \to |f\rangle$ asymptotic in-/out-states as $t \to \pm\infty$
  $|i\rangle$, $|f\rangle$ are separated, freely traveling single-particle states

Revision 12 (19.11.2014)

- S-matrix $\leftrightarrow$ residues of on-shell correlation functions

$$\prod_{k=1}^n \int_{\mathbb{R}^4} \mathrm{d}^4y_k\, e^{ip_ky_k} \prod_{l=1}^r \int_{\mathbb{R}^4} \mathrm{d}^4x_l\, e^{iq_lx_l} \left\langle \Omega \,\middle|\, T\prod_k \phi(y_k)\prod_l \phi(x_l) \,\middle|\, \Omega \right\rangle$$
$$= \prod_{k=1}^n \frac{i\sqrt{Z}}{p_k^2-m^2} \prod_{l=1}^r \frac{i\sqrt{Z}}{q_l^2-m^2}\,\langle p_1...p_n\;|S|\;q_1...q_r\rangle|_{\text{connected}}$$

- General aim in quantum field theory is to compute correlation functions of the form $\langle\Omega|T\prod_{i=1}^n\phi(x_i)|\Omega\rangle$[2]

Revision 13 (24.11.2014)

- Relate $\phi(x)$ to "free field" $\phi_{\mathrm{I}}(x)$ in the interaction picture

[2] Later also for excited states, not only the vacuum.

- $\phi(t,\boldsymbol{x}) = U^\dagger(t,t_0)\phi_\mathrm{I}(t,\boldsymbol{x})U(t,t_0)$
  where

$$U(t,t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)} = Te^{-i\int_{t_0}^{t} H_\mathrm{I}(t')\,\mathrm{d}t'},$$
$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^{t} \mathrm{d}t_1 ... \int_{t_0}^{t} \mathrm{d}t_n T H_\mathrm{I}(t_1)...H_\mathrm{I}(t_n)$$

- Property: $(\partial^2 + m_0^2)\phi_\mathrm{I} = 0$, $a_\mathrm{I}(\boldsymbol{p})|0\rangle = 0$ "free field", where $|0\rangle$: free vacuum
- Relate $|\Omega\rangle$ to $|0\rangle$ on which $a_\mathrm{I}(\boldsymbol{p})$ and $a_\mathrm{I}^\dagger(\boldsymbol{p})$ act:
  $|\Omega\rangle = \lim\limits_{T\to\infty(1-i\epsilon)} \dfrac{e^{-iHT}}{e^{-iE_\Omega T}\langle\Omega|0\rangle}|0\rangle$, where $H|\Omega\rangle = E_\Omega|\Omega\rangle$ with respect to the energy gauge $H_0|0\rangle = 0$

Revision 14 (26.11.2014)

- Correlation function in the full interaction theory $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_\mathrm{int}$

$$\langle\Omega|T\prod_{i=1}^{n}\phi(x_i)|\Omega\rangle = \frac{\left\langle 0 \,\middle|\, T\prod_{i=1}^{n}\phi_\mathrm{I}(x_i)e^{i\int_{\mathbb{R}^4}\mathrm{d}^4x\mathcal{L}_\mathrm{int}} \,\middle|\, 0\right\rangle}{\left\langle 0 \,\middle|\, Te^{i\int_{\mathbb{R}^4}\mathrm{d}^4x\mathcal{L}_\mathrm{int}} \,\middle|\, 0\right\rangle}$$

- Wick's theorem
  $T\phi(x_1)...\phi(x_N) = :\phi(x_1)...\phi(x_N) + \text{all possible contractions of operator pairs}:$,
  where $\langle 0|:\mathcal{O}:|0\rangle = 0$ except when $\mathcal{O} = c\cdot\mathbb{1}$, $c\in\mathbb{C}$
- $\langle 0|T\phi(x_1)...\phi(x_{2N})|0\rangle = D_\mathrm{F}(x_1-x_2)D_\mathrm{F}(x_3-x_4)...D_\mathrm{F}(x_{2N-1}-x_{2N})$
  $+$ all other contractions
  with $D_\mathrm{F}(x-y) = \displaystyle\int_{\mathbb{R}^4} \frac{\mathrm{d}^4p}{(2\pi)^4}\,\frac{i}{p^2-m^2+i\epsilon}\,e^{-ip(x-y)}$
  $\langle T{:}\phi^4(x){::}\phi^4(x){:}\phi(y_1)...\phi(y_n)\rangle$

Revision 15 (1.12.2014)

- $\lim\limits_{T\to\infty(1-i\epsilon)}\langle 0|T\phi(x)\phi(y)e^{-i\int_{-T}^{T}\mathrm{d}tH_\mathrm{I}(t)}|0\rangle = \sum\text{connected diagrams}\cdot e^{\sum\text{disconnected diagrams}}$,
  where ($\sum$ connected diagrams) is illustrated in Figure 4 and ($\sum$ disconnected diagrams) in Figure 5.

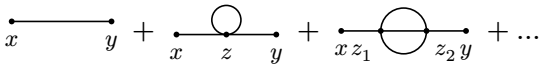


Figure 4: Sum of all at least partially connected diagrams with n external points

- $\lim\limits_{T\to\infty(1-i\epsilon)}\langle 0|Te^{-i\int_{-T}^{T}\mathrm{d}tH_\mathrm{I}(t)}|0\rangle = e^{\sum\text{disconnected diagrams}}$

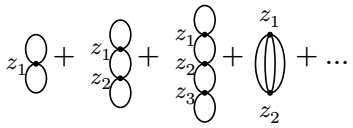


Figure 5: Sum of all entirely disconnected diagrams without external points

- such that

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\phi(x)\phi(y)e^{-i\int_{-T}^{T}\mathrm{d}tH_\mathrm{I}(t)}|0\rangle}{\langle 0|Te^{-i\int_{-T}^{T}\mathrm{d}tH_\mathrm{I}(t)}|0\rangle}$$
$$= \sum\text{connected diagrams}$$

$= \quad x \quad y \quad + \quad x \quad z \quad y \quad + \quad x\, z_1 \quad z_2\, y \quad + \ldots$

- more generally $\langle\Omega|T\prod_{i=1}^{n}\phi(x_i)|\Omega\rangle = \sum_{\substack{\text{over all (partially) connected}\\ \text{diagrams with } n \text{ external points}}}$

Revision 16 (3.12.2014)

- $\langle p_1 ... p_n \ |S|\ q_1 ... q_r\rangle|_{\text{connected}}$ (amputated correlation functions)
- $\langle f|S|i\rangle = \delta_{fi} + i(2\pi)^4\delta^{(4)}(p_f - p_i)\mathcal{M}_{fi}$
- $\omega_{fi} = \dfrac{P_{|i\rangle\to|f\rangle}}{\text{vol}_{\mathbb{R}}\cdot\text{time}}$ (transition rate)
- $\omega_{fi} = \dfrac{1}{N!}\prod_{n=1}^{N}\int_{\mathbb{R}^3}\dfrac{\mathrm{d}^3k}{(2\pi)^3}\dfrac{1}{2E_n}(2\pi)^4\delta^4\left(\sum_k p_k - \sum_l q_l\right)\ |\mathcal{M}_{fi}|^2$

Revision 17 (8.12.2014)

- No revision for this lecture.

Revision 18 (10.12.2014)

- Lorentz transformation laws for fields
  $x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu$, where $\mu,\nu \in \{0,1,2,3\}$
  $\phi^a(x) \mapsto \phi'^a(x) = R^a{}_b(\Lambda)\phi^b(\Lambda^{-1}x)$, where $a,b \in \{1,2,...,n\}$
  - scalar: $R(\Lambda) = 1$ (trivial representation, spin 0 particles)
  - vector: $R^\mu{}_\nu(\Lambda) = \Lambda^\mu{}_\nu$ (vector representation, spin 1, in the special case of the fundamental representation, we have $a,b = \mu,\nu$)
    $\Lambda^\mu{}_\nu = \left(e^{-\frac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma}}\right)^\mu{}_\nu$,
    where $(J^{\rho\sigma})^{\mu\nu} = i(\eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\rho\nu}\eta^{\sigma\mu})$
    and $[J^{\mu\nu}, J^{\rho\sigma}] = i(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\sigma}J^{\nu\rho})$
    E.g. spatial rotation by angle α around axis $\boldsymbol{n}$:

$$\omega_{ij} = \alpha\epsilon_{ijk}n^k, \text{which for } n = (1,0,0)^\dagger \text{ gives } \omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & -\alpha & 0 \end{pmatrix}$$

$$\text{and the corresponding rotation matrix } \Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

  - spinor representation? we look for fields that transform as
    $\Psi^A(x) \mapsto \Psi'^A(x) = \left(\Lambda_{1/2}\right)^A{}_B\Psi^B(\Lambda^{-1}x)$, where $\left(\Lambda_{1/2}\right)^A{}_B = \left(e^{-\frac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma}}\right)^A{}_B$
    representing spin $\frac{1}{2}$ ($\mu,\nu \in \{0,1,2,3\}$ and $A,B \in \{1,2,...,n\}$)
    answer: $(S^{\mu\nu})^A{}_B = \frac{i}{4}[\gamma^\mu,\gamma^\nu]^A{}_B$, where γ^μ are $n\times n$-matrices satisfying $\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}$

Revision 19 (15.12.2014)

- Spin-1/2-fields
  $\Psi^\dagger = (\Psi^*)^T$, $\Psi^\dagger\Psi$ not a Lorentz scalar because $\Lambda^\dagger_{\frac{1}{2}} \neq \Lambda^{-1}_{\frac{1}{2}}$
  γ^μ, $\mu \in \{0,1,2,3\}$
  $(\gamma^0)^\dagger = \gamma^0$, $(\gamma^i)^\dagger = -\gamma^i$

$\gamma^0 \Lambda_{\frac{1}{2}}^\dagger \gamma^0 = \Lambda_{\frac{1}{2}}^{-1}$

$\boxed{\bar{\Psi} = \Psi^\dagger \gamma^0}$ Dirac conjugate spinor

$|m| \mathrel{\hat{=}}$ mass of Ψ

$\boxed{S = \int_{\mathbb{R}^4} \mathrm{d}^4 x \, \bar{\Psi}\big(i\gamma^\mu \partial_\mu - m\big)\Psi}$ Dirac action

$\big(i\gamma^\mu \partial_\mu - m\big)\Psi = 0$ Dirac equation

Revision 20 (15.12.2014)

No revision.

Revision 21 (7.1.2015)

- Start from classical Lagrangian
  $S = \int_{\mathbb{R}^4} \mathrm{d}^4 x \, \bar{\Psi}\big(i\gamma^\mu \partial_\mu - m\big)\Psi$
  $\Pi_A = \dfrac{\partial \mathcal{L}}{\partial \dot{\Psi}^A} = i\Psi_A^\dagger$, $A \in \{1,2,3,4\}$ spinor indices
  and impose canonical *anti*commutation relations
  $\big\{\Psi^A(\boldsymbol{x}), \Psi_B^\dagger(\boldsymbol{y})\big\} = \delta^A{}_B \delta^{(3)}(\boldsymbol{x} - \boldsymbol{y})$
  $\big\{\Psi^A(\boldsymbol{x}), \Psi^B(\boldsymbol{y})\big\} = 0 = \big\{\Psi_A^\dagger(\boldsymbol{x}), \Psi_B^\dagger(\boldsymbol{y})\big\}$
  with the anticommutator $\{M, N\} = MN + NM = \{N, M\}$.
  For mode expansion
  $$\Psi(\boldsymbol{x}) = \sum_{s=\pm\frac{1}{2}} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \big(a_s(\boldsymbol{p}) u_s(\boldsymbol{p}) e^{i\boldsymbol{p}\boldsymbol{x}} + b_s^\dagger(\boldsymbol{p}) v_s(\boldsymbol{p}) e^{-i\boldsymbol{p}\boldsymbol{x}}\big)$$
  where $u_s(\boldsymbol{p})$ and $v_s(\boldsymbol{p})$ are spinor solutions to
  $(\gamma \cdot p - m) u_s(\boldsymbol{p}) = 0$, $(\gamma \cdot p + m) v_s(\boldsymbol{p}) = 0$
  We have $\big\{a_s(\boldsymbol{p}), a_r^\dagger(\boldsymbol{q})\big\} = (2\pi)^3 \delta_{sr} \delta^3(\boldsymbol{p} - \boldsymbol{q}) = \big\{b_s(\boldsymbol{p}), b_r^\dagger(\boldsymbol{q})\big\}$, while all other commutators vanish.
- Hamiltonian: $H = \displaystyle\int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} E_{\boldsymbol{p}} \sum_{s=\pm\frac{1}{2}} \big(a_s^\dagger(\boldsymbol{p}) a_s(\boldsymbol{p}) + b_s^\dagger(\boldsymbol{p}) b_s(\boldsymbol{p})\big)$, (after dropping vacuum energy)

Revision 22 (12.1.2015)

- Anticommutator
  $S^A{}_B(x - y) = \big\{\Psi^A(x), \bar{\Psi}_B(y)\big\}$
  $$\begin{aligned} S(x-y) &= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} \big[(\gamma \cdot p + m_0) e^{-ip(x-y)} + (\gamma \cdot p - m_0) e^{ip(x-y)}\big] \\ &= (i\gamma \cdot \partial + m_0)\big[D^{(0)}(x-y) - D^{(0)}(y-x)\big] \\ D^{(0)}(x-y) &= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} e^{-ip(x-y)} \quad \text{free scalar propagator} \end{aligned}$$
- Feynman propagator $S_{\mathrm{F}}(x - y) = \langle 0 | T\Psi(x)\bar{\Psi}(y) | 0 \rangle$
  Time-ordering symbol $T\Psi(x)\bar{\Psi}(y) = \begin{cases} \Psi(x)\bar{\Psi}(y), & \text{if } x^0 \geq y^0 \\ -\bar{\Psi}(y)\Psi(x), & \text{if } y^0 > x^0 \end{cases}$

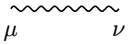
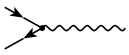
Revision 23 (14.1.2015)

No revision.

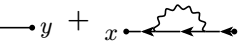
Revision 24 (19.1.2015)

- Free Maxwell action
  $S = \int_{\mathbb{R}^4} \mathrm{d}^4x \left(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu}\right)$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
  gauge invariance $A_\mu \leftarrow A_\mu + \partial_\mu \alpha$
- If Lorenz condition $\partial \cdot A = \partial_\mu A^\mu = 0$ imposed, then residual gauge symmetry remains $A_\mu \leftarrow A_\mu + \partial_\mu \alpha$ for $\Box \alpha = 0$
- Quantisation: start from gauge-fixed Lagrangian $\mathcal{L} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu$ together with constraint $\partial_\mu A^\mu = 0$
- $\Pi_\mu(x) = -\dot{A}_\mu(x)$, where $A_\mu(x) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{\sqrt{2\ |\boldsymbol{p}|}} \sum_{\lambda=0}^{3} \epsilon^\mu(\boldsymbol{p},\lambda)\left[a_\lambda(\boldsymbol{p})e^{-ipx} + a_\lambda^\dagger(\boldsymbol{p})e^{ipx}\right]$
  and $\epsilon^\mu(\boldsymbol{p},\lambda)\epsilon_\mu(\boldsymbol{p},\lambda') = \eta_{\lambda\lambda'}$
- $\left[a_\lambda(\boldsymbol{p}), a_{\lambda'}^\dagger(\boldsymbol{q})\right] = -\eta_{\lambda\lambda'}(2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})$
- $H = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3p}{(2\pi)^3}\ |\boldsymbol{p}|\ \left(\sum_{i=1}^{3} a_i^\dagger(\boldsymbol{p})a_i(\boldsymbol{p}) - a_0^\dagger(\boldsymbol{p})a_0(\boldsymbol{p})\right)$
- $|\boldsymbol{p},\lambda\rangle = \sqrt{2E_{\boldsymbol{p}}}a_\lambda^\dagger(\boldsymbol{p})|0\rangle$, $E_{\boldsymbol{p}} = |\boldsymbol{p}|$

Revision 25 (21.1.2015)

- QED Lagrangian $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{2}\left(\partial_\mu A^\mu\right)^2 + \bar{\Psi}\left(i\gamma^\mu D_\mu - m\right)\Psi$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu = \partial_\mu + ieA_\mu$, $\lambda = 1$ is called Feynman gauge
- photon propagator  $\hat{=} \frac{-i\eta_{\mu\nu}}{p^2 + i\epsilon}$
- Fermion propagator $\hat{=} \frac{i(\gamma \cdot p + m_0)}{p^2 - m_0^2 + i\epsilon}$
- interaction vertex  $\hat{=} -ie\gamma^\mu$

Revision 26 (26.1.2015)

- 1-loop electron propagator in QED
  $\langle\Omega|T\Psi(x)\bar{\Psi}(y)|\Omega\rangle =$  + ... (second diagram: self-energy at 1-loop)
  self energy: $-i\Sigma_2(\not{p}) = (-ie)^2 \int_{\mathbb{R}^4} \frac{\mathrm{d}^4k}{(2\pi)^4} \gamma^\mu \frac{i(\not{k} + m_0)}{k^2 - m_0^2 + i\epsilon} \gamma_\mu \frac{-i}{(p-k)^2 - \mu^2 + i\epsilon}$

Revision 27 (28.1.2015)

- regularised electron mass at 1-loop (order $\mathcal{O}(\alpha^2)$)

$$m - m_0 = \Sigma_2(\not{p} = m_0) + \mathcal{O}(\alpha^2) = \frac{\alpha_0}{2\pi} m_0 \int_0^1 \mathrm{d}x\,(2-x)\log\left(\frac{x\Lambda^2}{(1-x)^2 m_0^2 + x\mu^2}\right)$$

  gives $m_0 = m_0(m,\Lambda)$ for the bare mass m_0 in terms of the measured physical mass m, and Λ; plug in everywhere for bare quantities expressions of the form $m_0(m,\Lambda)$, etc., then Λ disappears for all physical quantities such as scattering amplitudes (this process is called renormalisation)
- Photon propagator (in Feynman gauge)

$$\left\langle\Omega\ |TA_\mu(x)A_\nu(y)|\ \Omega\right\rangle = \int_{\mathbb{R}^4} \frac{\mathrm{d}^4q}{(2\pi)^4} e^{-iq(x-y)} \left(\frac{-i}{q^2(1-\Pi(q^2))}\left(\eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \frac{-i}{q^2}\frac{q_\mu q_\nu}{q^2}\right)$$