Quantum Field Theory I

Lecture Revisions

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Revision 1 (13.10.2014)

Lecturer: Prof. Jürgen Berges
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Literature:

- Lecture notes "Quantum Field Theory I & II" by Timo Weigand, Chapters 1 to 6
- "An Introduction to Quantum Field Theory" by Peskin and Schröder

Tutorials: organized Dr. Viraf Mehta (v.mehta@thphys.uni-heidelberg.de), online registration
The course is passed by successfully standing the written examination at the end of the semester.
Plan:

1. The free scalar field
2. The interacting scalar field
3. Quantizing spin-$\frac{1}{2}$-fields
4. Quantizing spin-1-fields
5. Quantum electrodynamics (QED)
6. Classical non-abelian gauge theory

Revision 2 (17.10.2014)

Why Quantum Field Theory?

We will focus on elementary particle physics, hence relativistic QFT. However, virtually the same methods play a role in nuclear, atomic and condensed matter physics. In any relativistic field theory, the particle number is not conserved; since $E^2 = c^2\boldsymbol{p}^2 + m^2c^4$, energy can always be converted into particles and vice versa. This requires a *multiparticle* framework different from quantum mechanics.
QFT is a change of perspective from quantum mechanics in the following regards:

- The fundamentla entities are not the particles but rather "the field" - an abstract object that "penetrates" spacetime.
- Particles are the "excitations" of the field.

Revision 3 (20.10.2014)

- Free scalar field theory ($\varphi^* = \varphi$): $S = \frac{1}{2}\int_{\mathbb{R}^3} \mathrm{d}^4x\ \left[\partial_\mu\varphi\partial^\mu\varphi - m^2\varphi^2\right]$
- Euler-Lagrange: $\frac{\partial\mathcal{L}}{\partial\varphi} = \partial_\mu \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\varphi\right)}$
- Symmetry: $\varphi \to \varphi + \epsilon\cdot\delta\varphi + \mathcal{O}(\epsilon^2)$ $\qquad$ $(\partial\varphi = \partial_\nu\varphi\left(x^\mu\right))$

  $\mathcal{L} \to \mathcal{L} + \epsilon\partial_\mu F^\mu + \mathcal{O}(\epsilon^2)$
- Noether current: $j^\mu = \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\varphi\right)}\delta\varphi - F^\mu$
- Conserved charge: $\partial_\mu j^\mu = 0$, $Q = \int_{\mathbb{R}^3}\mathrm{d}^3x\, j^0$

Revision 4 (22.10.2014)

- Energy-momentum tensor: $T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial^\nu\varphi - \eta^{\mu\nu}\mathcal{L}$

  For this tensor, one has $\partial_\mu T^{\mu\nu} = 0$, $E = \int_{\mathbb{R}^3}\mathrm{d}^3x\, T^{00}$, and $p^i = \int_{\mathbb{R}^3}\mathrm{d}^3x\, T^{0i}$.

  Free scalar-field: $E = \int_{\mathbb{R}^3}\mathrm{d}^3x\ \left[\frac{1}{2}\dot\varphi^2 + \frac{1}{2}\left(\boldsymbol{\nabla}\varphi\right)^2 + \frac{1}{2}m^2\varphi^2\right]$, $\boldsymbol{p} = -\int_{\mathbb{R}^3}\mathrm{d}^3x\ \left(\dot\varphi\boldsymbol{\nabla}\varphi\right)$
- Canonical quantization

  i) Conjugate momentum density: $\pi\left(\boldsymbol{x},t\right) = \frac{\partial\mathcal{L}}{\partial\left(\dot\varphi\left(\boldsymbol{x},t\right)\right)}$

  ii) $H = \int_{\mathbb{R}^3}\mathrm{d}^3x\,\mathcal{H} = \int_{\mathbb{R}^3}\mathrm{d}^3x\ \left[\pi\dot\varphi - \mathcal{L}\right] \overset{\text{free theory}}{=} \int_{\mathbb{R}^3}\mathrm{d}^3x\ \left[\frac{1}{2}\dot\varphi^2 + \frac{1}{2}\left(\boldsymbol{\nabla}\varphi\right)^2 + \frac{1}{2}m^2\varphi^2\right]$

  iii) Promote $\varphi\left(\boldsymbol{x}\right)$ and $\pi\left(\boldsymbol{x}\right)$ to operators:
  - $\left[\varphi\left(\boldsymbol{x}\right), \pi\left(\boldsymbol{y}\right)\right] = i\delta^{(3)}\left(\boldsymbol{x}-\boldsymbol{y}\right)$
  - $\left[\varphi\left(\boldsymbol{x}\right), \varphi\left(\boldsymbol{y}\right)\right] = \left[\pi\left(\boldsymbol{x}\right), \pi\left(\boldsymbol{y}\right)\right] = 0$

Revision 5 (27.10.2014)

- Mode expansion:

  $\varphi\left(\boldsymbol{x}\right) = \int_{\mathbb{R}^3}\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\left(a\left(\boldsymbol{p}\right)e^{i\boldsymbol{px}} + a^\dagger\left(\boldsymbol{p}\right)e^{-i\boldsymbol{px}}\right)$

  $\pi\left(\boldsymbol{x}\right) = \int_{\mathbb{R}^3}\frac{\mathrm{d}^3p}{(2\pi)^3}\ (-i)\sqrt{\frac{\omega_{\boldsymbol{p}}}{2}}\left(a\left(\boldsymbol{p}\right)e^{i\boldsymbol{px}} - a^\dagger\left(\boldsymbol{p}\right)e^{-i\boldsymbol{px}}\right)$
- $\left[a\left(\boldsymbol{p}\right), a^\dagger\left(\boldsymbol{q}\right)\right] = (2\pi)^3\,\delta^{(3)}\left(\boldsymbol{p}-\boldsymbol{q}\right)$

  $\left[a\left(\boldsymbol{p}\right), a\left(\boldsymbol{q}\right)\right] = \left[a^\dagger\left(\boldsymbol{p}\right), a^\dagger\left(\boldsymbol{q}\right)\right] = 0 \quad \forall\boldsymbol{p},\boldsymbol{q}$

- 4-momentum: $P^\mu = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} p^\mu a^\dagger(\boldsymbol{p}) a(\boldsymbol{p})$, where $p^\mu = \left(p^0, \boldsymbol{p}\right)$ and $p^0 = \omega_{\boldsymbol{p}} = E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$
  $[P^\mu, a(\boldsymbol{p})] = -p^\mu a(\boldsymbol{p})$
  $\left[P^\mu, a^\dagger(\boldsymbol{p})\right] = p^\mu a^\dagger(\boldsymbol{p})$
- Vacuum: $a(\boldsymbol{p})|0\rangle = 0 \quad \boldsymbol{p}$
  $a^\dagger(\boldsymbol{p})$ creates 1-particle state with momentum $\boldsymbol{p}$: $P^\mu a^\dagger(\boldsymbol{p})|0\rangle = p^\mu a^\dagger(\boldsymbol{p})|0\rangle$
  N-particle state $a^\dagger(\boldsymbol{p}_1) \ldots a^\dagger(\boldsymbol{p}_N)|0\rangle$ with energy $E = \sum_{i=1}^N E_{\boldsymbol{p},i}$ and momentum $\boldsymbol{p} = \sum_{i=1}^N \boldsymbol{p}_i$.
  $\Rightarrow$ QFT is a multi-particle framework

Revision 6 (29.10.2014)

- N-particle momentum eigenstates
  $|\boldsymbol{p}_1, \ldots, \boldsymbol{p}_N\rangle = \prod_{i=1}^N \sqrt{2\omega_{\boldsymbol{p}_i}} a^\dagger(\boldsymbol{p}_i)|0\rangle$
  $\langle \boldsymbol{q}|\boldsymbol{p}\rangle = (2\pi)^3 2\omega_{\boldsymbol{p}} \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q})$
- Position eigenstates
  $|\boldsymbol{x}\rangle = \varphi(\boldsymbol{x})|0\rangle = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2\omega_{\boldsymbol{p}}} e^{-i\boldsymbol{p}\boldsymbol{x}} |\boldsymbol{p}\rangle$[1]
- Renormalization $H = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} \omega_{\boldsymbol{p}} a^\dagger(\boldsymbol{p}) a(\boldsymbol{p}) + \Delta_H$,
  where $\epsilon_0 = \frac{\Delta_H}{\mathrm{Vol}_{\mathbb{R}^3}} = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{\omega_{\boldsymbol{p}}}{2} \to \infty$ (UV divergent)
  To renormalize, we absorb ϵ_0 into V_0 in the Lagrangian $\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2} m^2 \varphi^2 - V_0$

Revision 7 (3.11.2014)

- Complex scalar field
  $\mathcal{L} = \partial_\mu \varphi^\dagger \partial^\mu \varphi - m^2 \varphi^\dagger \varphi$
  $\pi(t, \boldsymbol{x}) = \dot{\varphi}^\dagger(t, \boldsymbol{x})$, $\pi^\dagger(t, \boldsymbol{x}) = \dot{\varphi}(t, \boldsymbol{x})$
  $[\varphi(\boldsymbol{x}), \pi(\boldsymbol{y})] = \left[\varphi^\dagger(\boldsymbol{x}), \pi^\dagger(\boldsymbol{y})\right] = 0$
- Mode expansion
  $\varphi(\boldsymbol{x}) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}} \left(a(\boldsymbol{p}) e^{i\boldsymbol{p}\boldsymbol{x}} + b^\dagger(\boldsymbol{p}) e^{-i\boldsymbol{p}\boldsymbol{x}}\right)$
  $a^\dagger(\boldsymbol{p})|0\rangle$: particle with momentum $\boldsymbol{p}$ and charge -1,
  $b^\dagger(\boldsymbol{p})|0\rangle$: particle with momentum $\boldsymbol{p}$ and charge 1

[1] All these mode expansions and creating a particle at position $\boldsymbol{x}$ as simple as $|\boldsymbol{x}\rangle = \varphi(\boldsymbol{x})|0\rangle$ are only valid in the free theory without interaction and therefore without any influence of particle creation on nearby particles.

- Heisenberg Picture
  $\varphi(t,\boldsymbol{x}) = \varphi^{(H)}(t,\boldsymbol{x}) = e^{iH^{(S)}t}\varphi^{(S)}(\boldsymbol{x})\, e^{-iH^{(S)}t}$
  equal-time commutation relations $\left[\varphi(t,\boldsymbol{x}),\pi(t,\boldsymbol{x})\right] = i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})$
  Klein-Gordon operator equation $\left(\partial^2 + m^2\right)\varphi(t,\boldsymbol{x}) = 0$

Revision 8 (5.11.2014)

- Propagator: 2-point correlation function
  $D(x-y) = \langle 0|\,\varphi(x)\,\varphi(y)\,|0\rangle \overset{\text{free scalar theory}}{=} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3p}{(2\pi)^3}\,\frac{1}{2E_{\boldsymbol{p}}}e^{-ip(x-y)}$
- Commutator
  $\Delta(x-y) = \left[\varphi(x),\varphi(y)\right] = D(x-y) - D(y-x)$

Revision 9 (10.11.2014)

- Feynman propagator
  $D_{\mathrm{F}}(x-y) = \langle 0|\,T\,\varphi(x)\,\varphi(y)\,|0\rangle = \begin{cases} \langle 0|\,\varphi(x)\,\varphi(y)\,|0\rangle & x^0 \geq y^0 \\ \langle 0|\,\varphi(y)\,\varphi(x)\,|0\rangle & x^0 < y^0 \end{cases}$
  - Free theory (without interaction) $D_{\mathrm{F}}(x-y) = \int_{\mathbb{R}^4} \frac{\mathrm{d}^4p}{(2\pi)^4}\,\frac{i}{p^2 - m^2 i\epsilon}e^{-ip(x-y)}$
    where the $i\epsilon$-term represents time-ordering

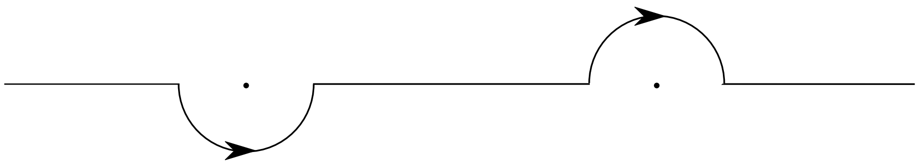


Figure 1: This integration path in the complex momentum plane yields the (time-ordered) Feynman propagator $D_{\mathrm{F}}(x-y)$.

- causally retarded and advanced propagators require different integration schemes
  - $D_{\mathrm{A}}(x-y)$

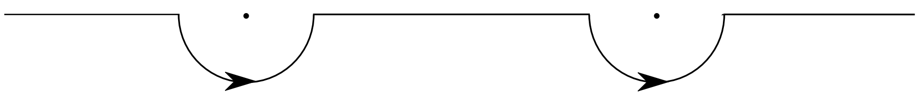


Figure 2: This integration path in the complex momentum plane yields the advanced propagator $D_{\mathrm{A}}(x-y)$.

  - $D_{\mathrm{R}}(x-y)$

Figure 3: This integration path in the complex momentum plane yields the retarded propagator $D_{\mathrm{R}}(x-y)$.

  - $D_{\mathrm{A}}(x-y)/D_{\mathrm{R}}(x-y)$ propagates information forward(backward in time
  - $D_{\mathrm{F}}(x-y)$ propagates positive frequency modes (e^{-ipx}) forward in time and negative frequency modes (e^{ipx}) backward in time

Revision 10 (12.11.2014)

Interacting scalar field theory

- $V(\varphi) = \frac{1}{2} m_0^2 \varphi^2 + \frac{1}{3!} g \varphi^3 + \frac{1}{4!} \lambda \varphi^4 + \ldots$
- $H \left| \lambda_{\boldsymbol{p}} \right\rangle = E_{\boldsymbol{p}}(\lambda) \left| \lambda_{\boldsymbol{p}} \right\rangle$
  $\boldsymbol{P} \left| \lambda_{\boldsymbol{p}} \right\rangle = \boldsymbol{p} \left| \lambda_{\boldsymbol{p}} \right\rangle$
  - The state parameter $\lambda_{\boldsymbol{p}}$ denotes 1-particle, multi-particle and bound states; it hides much of the complexity that comes with interaction.
- $\mathbb{1} = |\Omega\rangle \langle\Omega| + \sum_\lambda \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}(\lambda)} \left| \lambda_{\boldsymbol{p}} \right\rangle \left\langle \lambda_{\boldsymbol{p}} \right|$
- $\langle\Omega| \varphi(x) \varphi(y) |\Omega\rangle = \sum_\lambda \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}(\lambda)} e^{-ip(x-y)} \underbrace{\left| \langle\Omega| \varphi(0) \left| \lambda_{\boldsymbol{0}} \right\rangle \right|^2}_{\text{field strength}}$

  Note: $2E_{\boldsymbol{p}}(\lambda)$ is no longer the simple relativistic energy-momentum relation $2E_{\boldsymbol{p}}(\lambda) \neq \sqrt{p^2 + m^2}$, more complicatd in the interacting theory

Revision 11 (17.11.2014)

- $\langle\Omega| T\varphi(x) \varphi(y) |\Omega\rangle = \int_0^\infty \frac{\mathrm{d}M^2}{2\pi} \rho\left(M^2\right) D_\mathrm{F}\left(x - y, M^2\right)$

  spectral density $\rho\left(M^2\right) = \sum_\lambda 2\pi\delta\left(M^2 - m_\lambda^2\right) \left| \langle\Omega \left| \varphi(0) \right| \lambda_0\rangle \right|^2$
- Lesson: the full propagator $\langle\Omega| T\varphi(x) \varphi(y) |\Omega\rangle$ yields the mass "m" as the first pole
- The propagator is a two-point correlation function.
- n-point correlcation function: $\langle\Omega| T\varphi(x_1) \varphi(x_2) \ldots \varphi(x_n) |\Omega\rangle$
- scattering $|i\rangle \to |f\rangle$ asymptotic in-/out-states as $t \to \pm\infty$
  $|i\rangle$, $|f\rangle$ are separated, freely traveling single-particle states

Revision 12 (19.11.2014)

- S-matrix $\leftrightarrow$ residues of on-shell correlation functions

$$\prod_{k=1}^n \int_{\mathbb{R}^3} \mathrm{d}^4 y_k \, e^{ip_k y_k} \prod_{l=1}^r \int_{\mathbb{R}^3} \mathrm{d}^4 x_l \, e^{iq_l x_l} \left\langle \Omega \middle| T \prod_k \varphi(y_k) \prod_l \varphi(x_l) \middle| \Omega \right\rangle$$
$$= \prod_{k=1}^n \frac{i\sqrt{Z}}{p_k^2 - m^2} \prod_{l=1}^r \frac{i\sqrt{Z}}{q_l^2 - m^2} \ \langle p_1 \ldots p_n \left| S \right| q_1 \ldots q_r\rangle|_\text{connected}$$

- General aim in quantum field theory is to compute correlation functions of the form $\langle\Omega| T \prod_{i=1}^n \varphi(x_i) |\Omega\rangle$[2]

[2]Later also for excited states, not only the vacuum.

Revision 13 (24.11.2014)

- Relate $\varphi(x)$ to "free field" $\varphi_{\mathrm{I}}(x)$ in the interaction picture
- $\varphi(t, \boldsymbol{x}) = U^{\dagger}(t, t_0)\, \varphi_{\mathrm{I}}(t, \boldsymbol{x})\, U(t, t_0)$
  where $U(t, t_0) = e^{iH_0(t-t_0)} e^{-iH(t-t_0)} = e^{-i\int_{t_0}^{t} H_{\mathrm{I}}(t')\,\mathrm{d}t'}$,
  $$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^{t} \mathrm{d}t_1 \ldots \int_{t_0}^{t} \mathrm{d}t_n\, T H_{\mathrm{I}}(t_1) \ldots H_{\mathrm{I}}(t_n)$$
- Property: $\left(\partial^2 + m_0^2\right) \varphi_{\mathrm{I}} = 0$, $a_{\mathrm{I}}(\boldsymbol{p}) |0\rangle = 0$ "free field", where $|0\rangle$: free vacuum
- Relate $|\Omega\rangle$ to $|0\rangle$ on which $a_{\mathrm{I}}(\boldsymbol{p})$ and $a_{\mathrm{I}}^{\dagger}(\boldsymbol{p})$ act:
  $|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \dfrac{e^{-iHT}}{e^{-iE_\Omega T} \langle \Omega | 0\rangle} |0\rangle$, where $H|\Omega\rangle = E_\Omega |\Omega\rangle$ with respect to the energy gauge $H_0|0\rangle = 0$

Revision 14 (26.11.2014)

- Correlation function in the full interaction theory $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\mathrm{int}}$
  $$\langle \Omega | T \prod_{i=1}^{n} \varphi(x_i) |\Omega\rangle = \frac{\left\langle \Omega \left| T \prod_{i=1}^{n} \varphi(x_i)\, e^{-i\int_{\mathbb{R}^3} \mathrm{d}^4x\, \mathcal{L}_{\mathrm{int}}} \right| \Omega \right\rangle}{\left\langle \Omega \left| T e^{-i\int_{\mathbb{R}^3} \mathrm{d}^4x\, \mathcal{L}_{\mathrm{int}}} \right| \Omega \right\rangle}$$
- Wick's theorem
  $T\varphi(x_1) \ldots \varphi(x_N) = :\varphi(x_1) \ldots \varphi(x_N) + \text{all possible contractions of operator pairs}:$,
  where $\langle 0|\ :\mathcal{O}:\ |0\rangle = 0$ except when $\mathcal{O} = c \cdot \mathbb{1}$, $c \in \mathbb{C}$
- $$\langle 0| T\varphi(x_1) \ldots \varphi(x_{2N}) |0\rangle = D_{\mathrm{F}}(x_1 - x_2)\, D_{\mathrm{F}}(x_3 - x_4) \ldots D_{\mathrm{F}}(x_{2N-1} - x_{2N}) + \text{all other contractions}$$
  with $D_{\mathrm{F}}(x - y) = \displaystyle\int_{\mathbb{R}^4} \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}$
  $\langle T\ :\varphi^4(x):\ :\varphi^4(x):\ \varphi(y_1) \ldots \varphi(y_n)\rangle$

Revision 15 (1.12.2014)

- $\lim_{T\to\infty(1-i\epsilon)} \langle 0| T\varphi(x)\,\varphi(y)\, e^{-i\int_{-T}^{T} \mathrm{d}t H_{\mathrm{I}}(t)} |0\rangle = \sum \text{connected diagrams} \cdot e^{\sum \text{disconnected diagrams}}$,
  where $\left(\sum \text{connected diagrams}\right)$ is illustrated in fig. 4 and $\left(\sum \text{disconnected diagrams}\right)$ in fig. 5.

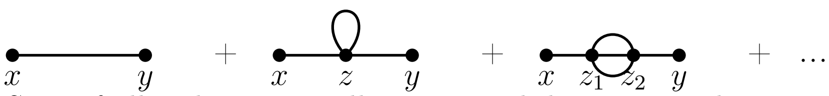


Figure 4: Sum of all at least partially connected diagrams with n external points

- $\lim_{T\to\infty(1-i\epsilon)} \langle 0| T e^{-i\int_{-T}^{T} \mathrm{d}t H_{\mathrm{I}}(t)} |0\rangle = e^{\sum \text{disconnected diagrams}}$

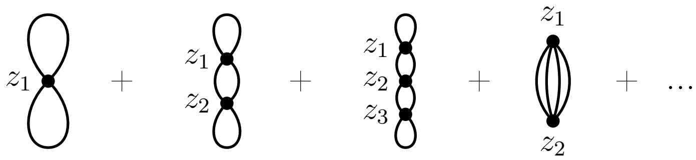
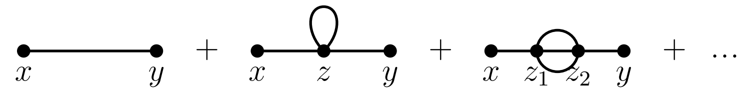


Figure 5: Sum of all entirely disconnected diagrams without external points

- such that $\langle\Omega|\, T\varphi(x)\,\varphi(y)\,|\Omega\rangle = \lim\limits_{T\to\infty(1-i\epsilon)} \dfrac{\langle 0|\, T\varphi(x)\,\varphi(y)\, e^{-i\int_{-T}^{T}\mathrm{d}t H_\mathrm{I}(t)}\,|0\rangle}{\langle 0|\, Te^{-i\int_{-T}^{T}\mathrm{d}t H_\mathrm{I}(t)}\,|0\rangle}$

  $= \sum$ connected diagrams

  $=$



- more generally $\langle\Omega|\, T\prod\limits_{i=1}^{n}\varphi(x_i)\,|\Omega\rangle = \sum\nolimits_{\text{diagrams with } n \text{ external points}}^{\text{over all (partially) connected}}$

Revision 16 (3.12.2014)

- $\langle p_1 \ldots p_n\, |S|\, q_1 \ldots q_r\rangle|_{\text{connected}}$ (amputated correlation functions)
- $\langle f\, |S|\, i\rangle = \delta_{fi} = i\,(2\pi)^4\,\delta^{(4)}(p_f - p_i)\,\mathcal{M}_{fi}$
- $\omega_{fi} = \dfrac{P_{|i\rangle\to|f\rangle}}{\mathrm{vol}_{\mathbb{R}}\cdot \text{time}}$ (transition rate)
- $\omega_{fi} = \dfrac{1}{N!}\prod\limits_{n=1}^{N}\int_{\mathbb{R}^3}\dfrac{\mathrm{d}^3k}{(2\pi)^3}\,\dfrac{1}{2E_n}\,(2\pi)^4\,\delta^4\left(\sum\limits_k p_k - \sum\limits_l q_l\right)|\mathcal{M}_{fi}|^2$

Revision 17 (8.12.2014)

- No Revision for this lecture.

Revision 18 (10.12.2014)

- Lorentz transformation laws for fields

  $x^\mu \longmapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu,$ where $\mu,\nu \in \{0,1,2,3\}$

  $\varphi^a(x) \longmapsto \varphi'^a(x) = R^a{}_b(\Lambda)\,\varphi^b(\Lambda^{-1}),$ where $a,b \in \{1,2,\ldots,n\}$

  - scalar: $R(\Lambda) = 1$ (trivial representation, spin 0 particles)
  - vector: $R^\mu{}_\nu(\Lambda) = \Lambda^\mu{}_\nu$ (vector representation, spin 1, in the special case of the fundamental representation, we have $a,b = \mu,\nu$)

    $\Lambda^\mu{}_\nu = \left(e^{-\frac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma}}\right)^\mu{}_\nu,$

    where $(J^{\rho\sigma})^{\mu\nu} = i\,(\eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\rho\nu}\eta^{\sigma\mu})$

    and $[J^{\mu\nu}, J^{\rho\sigma}] = i\,(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\sigma}J^{\nu\rho})$

    E.g. spatial rotation by angle α around axis $\boldsymbol{n}$:

$\omega_{ij} = i\epsilon_{ijk} n^k$, which for $n = (1,0,0)^\intercal$ gives $\omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & -\alpha & 0 \end{pmatrix}$

and the coresponding rotation matrix $\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix}$

- spinor representation? we look for fields that transform as

  $\Psi^A(x) \longmapsto \Psi'^A(x) = \left(\Lambda_{1/2}\right)^A{}_B \Psi^B\left(\Lambda^{-1}x\right)$, where $\left(\Lambda_{1/2}\right)^A{}_B = \left(e^{-\frac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma}}\right)^A{}_B$

  representing spin $\frac{1}{2}$ ($\mu, \nu \in \{0,1,2,3\}$ and $A, B \in \{1, 2, \dots, n\}$)

  answer: $(S^{\mu\nu})^A{}_B = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right]^A{}_B$, where γ^μ are $n \times n$-matrices satisfying $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$

Revision 19 (15.12.2014)

- Spin-1/2-fields

  $\Psi^\dagger = (\Psi^*)^\dagger$, $\Psi^\dagger\Psi$ not a Lorentz scalar because $\Lambda^\dagger_{\frac{1}{2}} \neq \Lambda^{-1}_{\frac{1}{2}}$

  γ^μ, $\mu \in \{0,1,2,3\}$

  $\left(\gamma^0\right)^\dagger = \gamma^0$, $\left(\gamma^i\right)^\dagger = -\gamma^i$

  $\gamma^0 \Lambda^\dagger_{\frac{1}{2}} \gamma^0 = \Lambda^{-1}_{\frac{1}{2}}$

  $\boxed{\bar{\Psi} = \Psi^\dagger \gamma^0}$ Dirac conjugate spinor

  $|m| \mathrel{\hat{=}}$ mass of Ψ

  $\boxed{S = \int_{\mathbb{R}^3} \mathrm{d}^4x\, \bar{\Psi}\left(i\gamma^\mu\partial_\mu - m\right)\Psi}$ Dirac action

  $\left(i\gamma^\mu\partial_\mu - m\right)\Psi$ Dirac equation

Revision 20 (15.12.2014)

No revision.

Revision 21 (7.1.2015)

- Start from classical Lagrangian

  $S = \int_{\mathbb{R}^3} \mathrm{d}^4x\, \bar{\Psi}\left(i\gamma^\mu\partial_\mu - m\right)\Psi$

  $\Pi_A = \frac{\partial\mathcal{L}}{\partial \dot{Psi}^A} = i\Psi^\dagger_A$, $A \in \{1,2,3,4\}$ spinor indices

  and impose canonical *anti*commutation relations

  $\left\{\Psi^A(\boldsymbol{x}), \Psi^\dagger_B(\boldsymbol{y})\right\} = \delta^A{}_B \delta^{(3)}(\boldsymbol{x} - \boldsymbol{y})$

$\{\Psi^A(\boldsymbol{x}), \Psi^B(\boldsymbol{y})\} = 0 = \{\Psi_A^\dagger(\boldsymbol{x}), \Psi_B^\dagger(\boldsymbol{y})\}$
with the anticommutator $\{M, N\} = MN + NM = \{N, M\}$.
For mode expansion

$$\Psi(\boldsymbol{x}) = \sum_{s=\pm\frac{1}{2}} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \left(a_s(\boldsymbol{p}) u_s(\boldsymbol{p}) e^{i\boldsymbol{px}} + b_s^\dagger(\boldsymbol{p}) v_s(\boldsymbol{p}) e^{-i\boldsymbol{px}} \right)$$

where $u_s(\boldsymbol{p})$ and $v_s(\boldsymbol{p})$ are spinor solutions to
$(\gamma \cdot p - m) u_s(\boldsymbol{p}) = 0$, $(\gamma \cdot p + m) v_s(\boldsymbol{p}) = 0$
We have $\{a_s(\boldsymbol{p}), a_r^\dagger(\boldsymbol{q})\} = (2\pi)^3 \delta_{sr} \delta^3(\boldsymbol{p} - \boldsymbol{q}) = \{b_s(\boldsymbol{p}), b_r^\dagger(\boldsymbol{q})\}$, while all other commutators vanish.

- Hamiltonian: $H = \int_{\mathbb{R}^3} \mathrm{d}^3 x \, E_{\boldsymbol{p}} \sum_{s=\pm\frac{1}{2}} \left(a_s^\dagger(\boldsymbol{p}) b_s(\boldsymbol{p}) + b_s^\dagger(\boldsymbol{p}) b_s(\boldsymbol{p}) \right)$, (after dropping vacuum energy)

Revision 22 (12.1.2015)

- Anticommutator
$S^A{}_B(x-y) = \{\Psi^A(x), \Psi_B(y)\}$

$$S(x-y) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} \left[(\gamma \cdot p - m_0) e^{-ip(x-y)} + (\gamma \cdot p + m_0) e^{ip(x-y)} \right]$$
$$= (\gamma \cdot \partial + m_0) \left[D^{(0)}(x-y) - D^{(0)}(y-x) \right]$$
$$D^{(0)}(x-y) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} e^{-ip(x-y)} \quad \text{free scalar propagator}$$

- Feynman propagator $S_{\mathrm{F}}(x-y) = \langle 0| T\Psi(x) \bar{\Psi}(y) |0\rangle$
Time-rodering symbol $T\Psi(x)\bar{\Psi}(y) = \begin{cases} \Psi(x)\bar{\Psi}(y), & \text{if } x^0 \geq y^0 \\ -\bar{\Psi}(y)\Psi(x), & \text{if } y^0 > x^0 \end{cases}$

Revision 23 (14.1.2015)

No revision.

Revision 24 (19.1.2015)

- Free Maxwell action
$$S = \int_{\mathbb{R}^3} \mathrm{d}^4 x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right), \quad \text{where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$
gauge invariance $A_\mu \leftarrow A_\mu + \partial_\mu \alpha$

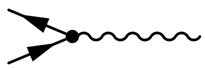
- If Lorenz condition $\partial \cdot A = \partial_\mu A^\mu = 0$ imposed, then residual gauge syymetry remains $A_\mu \leftarrow A_\mu + \partial_\mu \alpha$ for $\Box \alpha = 0$

- Quantisation: start from gauge-fixed Lagrangian $\mathcal{L} = -\frac{1}{2} \partial_\mu A_\nu \partial^\mu A^\nu$ together with constraint $\partial_\mu A^\mu = 0$

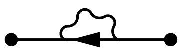
- $\Pi_\mu(x) = -\dot{A}_\mu(x)$, where $A_\mu(x) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\,|\boldsymbol{p}|}} \sum_{\lambda=0}^{3} \epsilon^\mu(\boldsymbol{p},\lambda) \left[a_\lambda(\boldsymbol{p})\, e^{-ipx} + a_\lambda^\dagger(\boldsymbol{p})\, e^{ipx}\right]$ and $\epsilon^\mu(\boldsymbol{p},\lambda)\,\epsilon_\mu(\boldsymbol{p},\lambda') = \eta_{\lambda\lambda'}$
- $\left[a_\lambda(\boldsymbol{p}), a_\lambda^\dagger(\boldsymbol{q})\right] = -\eta_{\lambda\lambda'}\,(2\pi)^3\,\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})$
- $H = \int_{\mathbb{R}^3} \mathrm{d}^3 x\ |\boldsymbol{p}| \left(\sum_{i=1}^{3} a_i^\dagger(\boldsymbol{p})\, a_i(\boldsymbol{p}) - a_0^\dagger(\boldsymbol{p})\, a_0(\boldsymbol{p})\right)$
- $|\boldsymbol{p},\lambda\rangle = \sqrt{2E_{\boldsymbol{p}}}\, a_\lambda^\dagger(\boldsymbol{p})\,|0\rangle$, $E_{\boldsymbol{p}} = |\boldsymbol{p}|$

Revision 25 (21.1.2015)

- QED Lagrangian $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{2}(\partial_\mu A^\mu)^2 + \bar{\Psi}\,(i\gamma^\mu \partial\mu - m)\,\Psi$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu = \partial_\mu + ieA_\mu$, $\lambda = 1$ is called Feynman gauge
- photon propagator $\hat{=} \frac{-i\eta_{\mu\nu}}{p^2 + i\epsilon}$

- Fermion propagator  $\hat{=} \frac{i(\gamma \cdot p + m_0)}{p^2 - m_0^2 + i\epsilon}$
- interaction vertex  $\hat{=} -ie\gamma^\mu$

Revision 26 (26.1.2015)

- 1-loop electron propagator in QED
  $\langle\Omega|\,T\Psi(x)\,\bar{\Psi}(y)\,|\Omega\rangle$ = x  y + x  y + ... (second diagram: self-energy at 1-loop)

  self energy: $-i\Sigma_2(\not{p}) = (-ie)^2 \int_{\mathbb{R}^4} \frac{\mathrm{d}^4 k}{(2\pi)^4}\, \gamma^\mu \frac{i(\not{k} + m_0)}{k^2 - m_0^2 + i\epsilon} \gamma_\mu \frac{-i}{(p-k)^2 - \mu^2 + i\epsilon}$

Revision 27 (28.1.2015)

- regularised electron mass at 1-loop (order $\mathcal{O}(\alpha^2)$)

  $$m - m_0 = \Sigma_2(\not{p} = m_0) + \mathcal{O}(\alpha^2) = \frac{\alpha_0}{2\pi} m_0 \int_0^1 \mathrm{d}x\,(2-x)\log\left(\frac{x\Lambda^2}{(1-x)^2 m_0^2 + x\mu^2}\right)$$

  gives $m_0 = m_0(m,\Lambda)$ for the bare mass m_0 in terms of the measured physical mass m, and Λ; plug in everywhere for bare quantities expressions of the form $m_0(m,\Lambda)$, etc., then Λ disappears for all physical quantities such as scattering amplitudes (this process is called renormalisation)
- Photon propagator (in Feynman gauge)

  $$\langle\Omega\,|TA_\mu(x)\,A^\nu(y)|\,\Omega\rangle = \int_{\mathbb{R}^4} \frac{\mathrm{d}^4 p}{(2\pi)^4}\, e^{-ip(x-y)} \left(\frac{-i}{q^2(1-\Pi(q^2))}\left(\eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \frac{-i}{q^2}\frac{q_\mu q_\nu}{q^2}\right)$$