

Quantum Field Theory I

Lecture Revisions

Janosh Riebesell, winter semester 2014/15

Revision 1 (13.10.2014)

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Literature:

- Lecture notes “Quantum Field Theory I & II” by Timo Weigand, Chapters 1 to 6
- “An Introduction to Quantum Field Theory” by Peskin and Schröder

Tutorials: organized by Dr. Viraf Mehta (v.mehta@thphys.uni-heidelberg.de), online registration
The course is passed by successfully standing the written examination at the end of the semester.

Plan:

1. The free scalar field
2. The interacting scalar field
3. Quantizing spin- $\frac{1}{2}$ -fields
4. Quantizing spin-1-fields
5. Quantum electrodynamics (QED)
6. Classical non-abelian gauge theory

Revision 2 (17.10.2014)

Why Quantum Field Theory?

We will focus on elementary particle physics, hence relativistic QFT. However, virtually the same methods play a role in nuclear, atomic and condensed matter physics. In any relativistic field theory, the particle number is not conserved; since $E^2 = c^2 \mathbf{p}^2 + m^2 c^4$, energy can always be converted into particles and vice versa. This requires a *multiparticle* framework different from quantum mechanics.

QFT is a change of perspective from quantum mechanics in the following regards:

- The fundamental entities are not the particles but rather “the field” – an abstract object that “penetrates” spacetime.
- Particles are the “excitations” of the field.

Revision 3 (20.10.2014)

- Free scalar field theory ($\phi^* = \phi$): $S = \frac{1}{2} \int_{\mathbb{R}^4} d^4x [\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2]$
- Euler-Lagrange: $\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)}$
- Symmetry: $\phi \rightarrow \phi + \epsilon \cdot \delta \phi + \mathcal{O}(\epsilon^2)$ ($\partial \phi = \partial_\nu \phi(x^\mu)$)
 $\mathcal{L} \rightarrow \mathcal{L} + \epsilon \partial_\mu F^\mu + \mathcal{O}(\epsilon^2)$
- Noether current: $j^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta \phi - F^\mu$
- Conserved charge: $\partial_\mu j^\mu = 0$, $Q = \int_{\mathbb{R}^3} d^3x j^0$

Revision 4 (22.10.2014)

- Energy-momentum tensor: $T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L}$

For this tensor, one has $\partial_\mu T^{\mu\nu} = 0$, $E = \int_{\mathbb{R}^3} d^3x T^{00}$, and $p^i = \int_{\mathbb{R}^3} d^3x T^{0i}$.

Free scalar field: $E = \int_{\mathbb{R}^3} d^3x \left[\frac{1}{2} \dot{\phi}^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 \right]$, $\mathbf{p} = - \int_{\mathbb{R}^3} d^3x (\dot{\phi} \nabla \phi)$

- Canonical quantization

i) Conjugate momentum density: $\pi(\mathbf{x}, t) = \frac{\partial \mathcal{L}}{\partial(\dot{\phi}(\mathbf{x}, t))}$

ii) $H = \int_{\mathbb{R}^3} d^3x \mathcal{H} = \int_{\mathbb{R}^3} d^3x [\pi \dot{\phi} - \mathcal{L}] \stackrel{\text{free theory}}{=} \int_{\mathbb{R}^3} d^3x \left[\frac{1}{2} \dot{\phi}^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 \right]$

iii) Promote $\phi(\mathbf{x})$ and $\pi(\mathbf{x})$ to operators:

- $[\phi(\mathbf{x}), \pi(\mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y})$
- $[\phi(\mathbf{x}), \phi(\mathbf{y})] = [\pi(\mathbf{x}), \pi(\mathbf{y})] = 0$

Revision 5 (27.10.2014)

- Mode expansion:

$$\phi(\mathbf{x}) = \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} (a(\mathbf{p})e^{i\mathbf{p}\mathbf{x}} + a^\dagger(\mathbf{p})e^{-i\mathbf{p}\mathbf{x}})$$

$$\pi(\mathbf{x}) = \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3} (-i)\sqrt{\frac{\omega_{\mathbf{p}}}{2}} (a(\mathbf{p})e^{i\mathbf{p}\mathbf{x}} - a^\dagger(\mathbf{p})e^{-i\mathbf{p}\mathbf{x}})$$

- $[a(\mathbf{p}), a^\dagger(\mathbf{q})] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q})$
 $[a(\mathbf{p}), a(\mathbf{q})] = [a^\dagger(\mathbf{p}), a^\dagger(\mathbf{q})] = 0 \quad \forall \mathbf{p}, \mathbf{q}$
- 4-momentum: $P^\mu = \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3} p^\mu a^\dagger(\mathbf{p})a(\mathbf{p})$, where $p^\mu = (p^0, \mathbf{p})$ and $p^0 = \omega_{\mathbf{p}} = E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$
 $[P^\mu, a(\mathbf{p})] = -p^\mu a(\mathbf{p})$
 $[P^\mu, a^\dagger(\mathbf{p})] = p^\mu a^\dagger(\mathbf{p})$
- Vacuum: $a(\mathbf{p})|0\rangle = 0 \quad \forall \mathbf{p}$
 $a^\dagger(\mathbf{p})$ creates 1-particle state with momentum $\mathbf{p}$: $P^\mu a^\dagger(\mathbf{p})|0\rangle = p^\mu a^\dagger(\mathbf{p})|0\rangle$
 N -particle state $a^\dagger(\mathbf{p}_1) \dots a^\dagger(\mathbf{p}_N)|0\rangle$ with energy $E = \sum_{i=1}^N E_{\mathbf{p}_i}$ and momentum $\mathbf{p} = \sum_{i=1}^N \mathbf{p}_i$.
 $\Rightarrow$ QFT is a multi-particle framework

Revision 6 (29.10.2014)

- N -particle momentum eigenstates
 $|\mathbf{p}_1, \dots, \mathbf{p}_N\rangle = \prod_{i=1}^N \sqrt{2\omega_{\mathbf{p}_i}} a^\dagger(\mathbf{p}_i)|0\rangle$
 $\langle \mathbf{q} | \mathbf{p} \rangle = (2\pi)^3 2\omega_{\mathbf{p}} \delta^{(3)}(\mathbf{p} - \mathbf{q})$
- Position eigenstates

$$|\mathbf{x}\rangle = \phi(\mathbf{x})|0\rangle = \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} e^{-i\mathbf{p}\mathbf{x}} |\mathbf{p}\rangle^1$$

¹All these mode expansions and creating a particle at position $\mathbf{x}$ as simple as $|\mathbf{x}\rangle = \phi(\mathbf{x})|0\rangle$ are only valid in the free theory without interaction and therefore without any influence of particle creation on nearby particles.

- Renormalization $H = \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3} \omega_{\mathbf{p}} a^\dagger(\mathbf{p}) a(\mathbf{p}) + \Delta_H$,

where $\epsilon_0 = \frac{\Delta_H}{\text{Vol}_{\mathbb{R}^3}} = \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3} \frac{\omega_{\mathbf{p}}}{2} \rightarrow \infty$ (UV divergent)

To renormalize, we absorb ϵ_0 into V_0 in the Lagrangian $\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - V_0$

Revision 7 (3.11.2014)

- Complex scalar field
 $\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi$
 $\pi(t, \mathbf{x}) = \dot{\phi}^\dagger(t, \mathbf{x}), \pi^\dagger(t, \mathbf{x}) = \dot{\phi}(t, \mathbf{x})$
 $[\phi(\mathbf{x}), \pi(\mathbf{y})] = [\phi^\dagger(\mathbf{x}), \pi^\dagger(\mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y})$, all other commutators vanish
- Mode expansion

$$\phi(\mathbf{x}) = \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} (a(\mathbf{p}) e^{i\mathbf{p}\mathbf{x}} + b^\dagger(\mathbf{p}) e^{-i\mathbf{p}\mathbf{x}})$$

 $a^\dagger(\mathbf{p})|0\rangle$: particle with momentum $\mathbf{p}$ and charge -1 ,
 $b^\dagger(\mathbf{p})|0\rangle$: particle with momentum $\mathbf{p}$ and charge 1
- Heisenberg Picture
 $\phi(t, \mathbf{x}) = \phi^{(H)}(t, \mathbf{x}) = e^{iH^{(S)}t} \phi^{(S)}(\mathbf{x}) e^{-iH^{(S)}t}$
equal-time commutation relations $[\phi(t, \mathbf{x}), \pi(t, \mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y})$
Klein-Gordon operator equation $(\partial^2 + m^2)\phi(t, \mathbf{x}) = 0$

Revision 8 (5.11.2014)

- Propagator: 2-point correlation function

$$D(x - y) = \langle 0 | \phi(x) \phi(y) | 0 \rangle \stackrel{\text{free scalar theory}}{=} \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{-ip(x-y)}$$
- Commutator
 $\Delta(x - y) = [\phi(x), \phi(y)] = D(x - y) - D(y - x)$

Revision 9 (10.11.2014)

- Feynman propagator

$$D_F(x - y) = \langle 0 | T \phi(x) \phi(y) | 0 \rangle = \begin{cases} \langle 0 | \phi(x) \phi(y) | 0 \rangle & x^0 \geq y^0 \\ \langle 0 | \phi(y) \phi(x) | 0 \rangle & x^0 < y^0 \end{cases}$$
- Free theory (without interaction) $D_F(x - y) = \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}$
where the $i\epsilon$ -term represents time-ordering

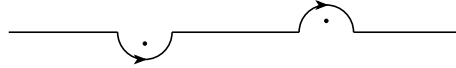


Figure 1: This integration path in the complex momentum plane yields the (time-ordered) Feynman propagator $D_F(x - y)$.

- causally retarded and advanced propagators require different integration schemes
- $D_A(x - y)$

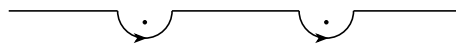


Figure 2: This integration path in the complex momentum plane yields the advanced propagator $D_A(x - y)$.

- $D_R(x - y)$

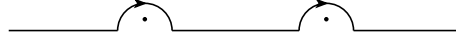


Figure 3: This integration path in the complex momentum plane yields the retarded propagator $D_R(x - y)$.

- $D_R(x - y)/D_A(x - y)$ propagates information forward (backward) in time
- $D_F(x - y)$ propagates positive frequency modes (e^{-ipx}) forward in time and negative frequency modes (e^{ipx}) backward in time

Revision 10 (12.11.2014)

Interacting scalar field theory

- $V(\phi) = \frac{1}{2} m_0^2 \phi^2 + \frac{1}{3!} g \phi^3 + \frac{1}{4!} \lambda \phi^4 + \dots$
- $H|\lambda_p\rangle = E_p(\lambda)|\lambda_p\rangle$
 $P|\lambda_p\rangle = p|\lambda_p\rangle$
 - The state parameter λ_p denotes 1-particle, multi-particle and bound states; it hides much of the complexity that comes with interaction.
- $\mathbb{1} = |\Omega\rangle\langle\Omega| + \sum_{\lambda} \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p(\lambda)} |\lambda_p\rangle\langle\lambda_p|$
- $\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \sum_{\lambda} \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p(\lambda)} e^{-ip(x-y)} \underbrace{|\langle\Omega|\phi(0)|\lambda_0\rangle|^2}_{\text{field strength}}$

Note: $E_p(\lambda) = \sqrt{p^2 + m_{\lambda}^2}$ involves the rest energy m_{λ} of the state λ (e.g. the invariant mass of a multi-particle or bound state) rather than a single particle mass m

Revision 11 (17.11.2014)

- $\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \int_0^{\infty} \frac{dM^2}{2\pi} \rho(M^2) D_F(x - y, M^2)$
spectral density $\rho(M^2) = \sum_{\lambda} 2\pi\delta(M^2 - m_{\lambda}^2) |\langle\Omega|\phi(0)|\lambda_0\rangle|^2$
- Lesson: the full propagator $\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle$ yields the mass “ m ” as the first pole
- The propagator is a two-point correlation function.
- n -point correlation function: $\langle\Omega|T\phi(x_1)\phi(x_2)\dots\phi(x_n)|\Omega\rangle$
- scattering $|i\rangle \rightarrow |f\rangle$ asymptotic in-/out-states as $t \rightarrow \pm\infty$
 $|i\rangle, |f\rangle$ are separated, freely traveling single-particle states

Revision 12 (19.11.2014)

- S -matrix $\leftrightarrow$ residues of on-shell correlation functions

$$\begin{aligned} & \prod_{k=1}^n \int_{\mathbb{R}^4} d^4y_k e^{ip_k y_k} \prod_{l=1}^r \int_{\mathbb{R}^4} d^4x_l e^{iq_l x_l} \left\langle \Omega \left| T \prod_k \phi(y_k) \prod_l \phi(x_l) \right| \Omega \right\rangle \\ &= \prod_{k=1}^n \frac{i\sqrt{Z}}{p_k^2 - m^2} \prod_{l=1}^r \frac{i\sqrt{Z}}{q_l^2 - m^2} \langle p_1 \dots p_n | S | q_1 \dots q_r \rangle_{\text{connected}} \end{aligned}$$

- General aim in quantum field theory is to compute correlation functions of the form $\langle\Omega|T \prod_{i=1}^n \phi(x_i)|\Omega\rangle^2$

Revision 13 (24.11.2014)

- Relate $\phi(x)$ to “free field” $\phi_I(x)$ in the interaction picture

²Later also for excited states, not only the vacuum.

- $\phi(t, \mathbf{x}) = U^\dagger(t, t_0)\phi_I(t, \mathbf{x})U(t, t_0)$
where

$$U(t, t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)} = T e^{-i \int_{t_0}^t H_I(t') dt'},$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^t dt_1 \dots \int_{t_0}^t dt_n T H_I(t_1) \dots H_I(t_n)$$

- Property: $(\partial^2 + m_0^2)\phi_I = 0$, $a_I(\mathbf{p})|0\rangle = 0$ “free field”, where $|0\rangle$: free vacuum
- Relate $|\Omega\rangle$ to $|0\rangle$ on which $a_I(\mathbf{p})$ and $a_I^\dagger(\mathbf{p})$ act:

$$|\Omega\rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{e^{-iHT}}{e^{-iE_\Omega T} \langle \Omega|0\rangle} |0\rangle, \quad \text{where } H|\Omega\rangle = E_\Omega|\Omega\rangle \text{ with respect to the energy gauge}$$

$$H_0|0\rangle = 0$$

Revision 14 (26.11.2014)

- Correlation function in the full interaction theory $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}$

$$\langle \Omega | T \prod_{i=1}^n \phi(x_i) | \Omega \rangle = \frac{\langle 0 | T \prod_{i=1}^n \phi_I(x_i) e^{i \int_{\mathbb{R}^4} d^4x \mathcal{L}_{\text{int}}} | 0 \rangle}{\langle 0 | T e^{i \int_{\mathbb{R}^4} d^4x \mathcal{L}_{\text{int}}} | 0 \rangle}$$

- Wick’s theorem
 $T\phi(x_1)\dots\phi(x_N) = :\phi(x_1)\dots\phi(x_N) + \text{all possible contractions of operator pairs} :$,
where $\langle 0 | \mathcal{O} | 0 \rangle = 0$ except when $\mathcal{O} = c \cdot \mathbb{1}$, $c \in \mathbb{C}$
- $\langle 0 | T\phi(x_1)\dots\phi(x_{2N}) | 0 \rangle = D_F(x_1 - x_2)D_F(x_3 - x_4)\dots D_F(x_{2N-1} - x_{2N})$
+ all other contractions

$$\text{with } D_F(x - y) = \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}$$

$$\langle T : \phi^4(x) : : \phi^4(x) : \phi(y_1) \dots \phi(y_n) \rangle$$

Revision 15 (1.12.2014)

- $\lim_{T \rightarrow \infty(1-i\epsilon)} \langle 0 | T\phi(x)\phi(y) e^{-i \int_{-T}^T dt H_I(t)} | 0 \rangle = \sum \text{connected diagrams} \cdot e^{\sum \text{disconnected diagrams}}$,
where $(\sum \text{connected diagrams})$ is illustrated in Figure 4 and $(\sum \text{disconnected diagrams})$ in Figure 5.

Figure 4: Sum of all at least partially connected diagrams with n external points

- $\lim_{T \rightarrow \infty(1-i\epsilon)} \langle 0 | T e^{-i \int_{-T}^T dt H_I(t)} | 0 \rangle = e^{\sum \text{disconnected diagrams}}$

Figure 5: Sum of all entirely disconnected diagrams without external points

- such that

$$\langle \Omega | T\phi(x)\phi(y) | \Omega \rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\langle 0 | T\phi(x)\phi(y) e^{-i \int_{-T}^T dt H_I(t)} | 0 \rangle}{\langle 0 | T e^{-i \int_{-T}^T dt H_I(t)} | 0 \rangle}$$

$$= \sum \text{connected diagrams}$$

$$= \begin{array}{c} \bullet \\ x \end{array} \text{---} \begin{array}{c} \bullet \\ y \end{array} + \begin{array}{c} \bullet \\ x \end{array} \text{---} \begin{array}{c} \circ \\ z \end{array} \text{---} \begin{array}{c} \bullet \\ y \end{array} + \begin{array}{c} \bullet \\ x \end{array} \text{---} \begin{array}{c} \bigcirc \\ z_1 \quad z_2 \end{array} \text{---} \begin{array}{c} \bullet \\ y \end{array} + \dots$$

- more generally $\langle \Omega | T \prod_{i=1}^n \phi(x_i) | \Omega \rangle = \sum_{\text{diagrams with } n \text{ external points}}^{\text{over all (partially) connected}}$

Revision 16 (3.12.2014)

- $\langle p_1 \dots p_n | S | q_1 \dots q_r \rangle_{\text{connected}}$ (amputated correlation functions)
- $\langle f | S | i \rangle = \delta_{fi} + i(2\pi)^4 \delta^{(4)}(p_f - p_i) \mathcal{M}_{fi}$
- $\omega_{fi} = \frac{P_{|i\rangle \rightarrow |f\rangle}}{\text{vol}_{\mathbb{R}} \cdot \text{time}}$ (transition rate)
- $\omega_{fi} = \frac{1}{N!} \prod_{n=1}^N \int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3} \frac{1}{2E_n} (2\pi)^4 \delta^4 \left(\sum_k p_k - \sum_l q_l \right) |\mathcal{M}_{fi}|^2$

Revision 17 (8.12.2014)

- No revision for this lecture.

Revision 18 (10.12.2014)

- Lorentz transformation laws for fields
 - $x^\mu \mapsto x'^\mu = \Lambda^\mu_\nu x^\nu$, where $\mu, \nu \in \{0, 1, 2, 3\}$
 - $\phi^a(x) \mapsto \phi'^a(x) = R^a_b(\Lambda) \phi^b(\Lambda^{-1}x)$, where $a, b \in \{1, 2, \dots, n\}$
 - scalar: $R(\Lambda) = 1$ (trivial representation, spin 0 particles)
 - vector: $R^\mu_\nu(\Lambda) = \Lambda^\mu_\nu$ (vector representation, spin 1, in the special case of the fundamental representation, we have $a, b = \mu, \nu$)
 - $\Lambda^\mu_\nu = \left(e^{-\frac{i}{2} \omega_{\rho\sigma} J^{\rho\sigma}} \right)^\mu_\nu$,
 - where $(J^{\rho\sigma})^{\mu\nu} = i(\eta^{\rho\mu} \eta^{\sigma\nu} - \eta^{\rho\nu} \eta^{\sigma\mu})$
 - and $[J^{\mu\nu}, J^{\rho\sigma}] = i(\eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\nu\sigma} J^{\mu\rho} + \eta^{\mu\sigma} J^{\nu\rho})$
 - E.g. spatial rotation by angle α around axis $\mathbf{n}$:

$$\omega_{ij} = \alpha \epsilon_{ijk} n^k, \text{ which for } n = (1, 0, 0)^\dagger \text{ gives } \omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & -\alpha & 0 \end{pmatrix}$$

$$\text{and the corresponding rotation matrix } \Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

- spinor representation? we look for fields that transform as
 - $\Psi^A(x) \mapsto \Psi'^A(x) = (\Lambda_{1/2})^A_B \Psi^B(\Lambda^{-1}x)$, where $(\Lambda_{1/2})^A_B = \left(e^{-\frac{i}{2} \omega_{\rho\sigma} J^{\rho\sigma}} \right)^A_B$
 - representing spin $\frac{1}{2}$ ($\mu, \nu \in \{0, 1, 2, 3\}$ and $A, B \in \{1, 2, \dots, n\}$)
 - answer: $(S^{\mu\nu})^A_B = \frac{i}{4} [\gamma^\mu, \gamma^\nu]^A_B$, where γ^μ are $n \times n$ -matrices satisfying $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$

Revision 19 (15.12.2014)

- Spin-1/2-fields
 - $\Psi^\dagger = (\Psi^*)^T$, $\Psi^\dagger \Psi$ not a Lorentz scalar because $\Lambda_{\frac{1}{2}}^\dagger \neq \Lambda_{\frac{1}{2}}^{-1}$
 - $\gamma^\mu, \mu \in \{0, 1, 2, 3\}$
 - $(\gamma^0)^\dagger = \gamma^0, (\gamma^i)^\dagger = -\gamma^i$

$$\gamma^0 \Lambda_{\frac{1}{2}}^\dagger \gamma^0 = \Lambda_{\frac{1}{2}}^{-1}$$

$$\boxed{\bar{\Psi} = \Psi^\dagger \gamma^0} \quad \text{Dirac conjugate spinor}$$

$$|m| \hat{=} \text{mass of } \Psi$$

$$\boxed{S = \int_{\mathbb{R}^4} d^4x \bar{\Psi} (i\gamma^\mu \partial_\mu - m) \Psi} \quad \text{Dirac action}$$

$$(i\gamma^\mu \partial_\mu - m) \Psi = 0 \quad \text{Dirac equation}$$

Revision 20 (15.12.2014)

No revision.

Revision 21 (7.1.2015)

- Start from classical Lagrangian

$$S = \int_{\mathbb{R}^4} d^4x \bar{\Psi} (i\gamma^\mu \partial_\mu - m) \Psi$$

$$\Pi_A = \frac{\partial \mathcal{L}}{\partial \dot{\Psi}^A} = i\Psi_A^\dagger, \quad A \in \{1, 2, 3, 4\} \text{ spinor indices}$$

and impose canonical *anticommutation* relations

$$\{\Psi^A(x), \Psi_B^\dagger(y)\} = \delta^A_B \delta^{(3)}(x - y)$$

$$\{\Psi^A(x), \Psi^B(y)\} = 0 = \{\Psi_A^\dagger(x), \Psi_B^\dagger(y)\}$$

with the anticommutator $\{M, N\} = MN + NM = \{N, M\}$.

For mode expansion

$$\Psi(x) = \sum_{s=\pm\frac{1}{2}} \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_s(\mathbf{p}) u_s(\mathbf{p}) e^{ipx} + b_s^\dagger(\mathbf{p}) v_s(\mathbf{p}) e^{-ipx})$$

where $u_s(\mathbf{p})$ and $v_s(\mathbf{p})$ are spinor solutions to

$$(\gamma \cdot p - m) u_s(\mathbf{p}) = 0, \quad (\gamma \cdot p + m) v_s(\mathbf{p}) = 0$$

We have $\{a_s(\mathbf{p}), a_r^\dagger(\mathbf{q})\} = (2\pi)^3 \delta_{sr} \delta^3(\mathbf{p} - \mathbf{q}) = \{b_s(\mathbf{p}), b_r^\dagger(\mathbf{q})\}$, while all other commutators vanish.

- Hamiltonian: $H = \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3} E_p \sum_{s=\pm\frac{1}{2}} (a_s^\dagger(\mathbf{p}) a_s(\mathbf{p}) + b_s^\dagger(\mathbf{p}) b_s(\mathbf{p}))$, (after dropping vacuum energy)

Revision 22 (12.1.2015)

- Anticommutator

$$S^A_B(x - y) = \{\Psi^A(x), \bar{\Psi}_B(y)\}$$

$$\begin{aligned} S(x - y) &= \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} [(\gamma \cdot p + m_0) e^{-ip(x-y)} + (\gamma \cdot p - m_0) e^{ip(x-y)}] \\ &= (i\gamma \cdot \partial + m_0) [D^{(0)}(x - y) - D^{(0)}(y - x)] \end{aligned}$$

$$D^{(0)}(x - y) = \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)} \quad \text{free scalar propagator}$$

- Feynman propagator $S_F(x - y) = \langle 0 | T \Psi(x) \bar{\Psi}(y) | 0 \rangle$

$$\text{Time-ordering symbol } T \Psi(x) \bar{\Psi}(y) = \begin{cases} \Psi(x) \bar{\Psi}(y), & \text{if } x^0 \geq y^0 \\ -\bar{\Psi}(y) \Psi(x), & \text{if } y^0 > x^0 \end{cases}$$

Revision 23 (14.1.2015)

No revision.

Revision 24 (19.1.2015)

- Free Maxwell action

$$S = \int_{\mathbb{R}^4} d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right), \quad \text{where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

gauge invariance $A_\mu \leftarrow A_\mu + \partial_\mu \alpha$

- If Lorenz condition $\partial \cdot A = \partial_\mu A^\mu = 0$ imposed, then residual gauge symmetry remains $A_\mu \leftarrow A_\mu + \partial_\mu \alpha$ for $\square \alpha = 0$
- Quantisation: start from gauge-fixed Lagrangian $\mathcal{L} = -\frac{1}{2} \partial_\mu A_\nu \partial^\mu A^\nu$ together with constraint $\partial_\mu A^\mu = 0$
- $\Pi_\mu(x) = -\dot{A}_\mu(x)$, where $A_\mu(x) = \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2}|\mathbf{p}|} \sum_{\lambda=0}^3 \epsilon^\mu(\mathbf{p}, \lambda) [a_\lambda(\mathbf{p}) e^{-ipx} + a_\lambda^\dagger(\mathbf{p}) e^{ipx}]$
and $\epsilon^\mu(\mathbf{p}, \lambda) \epsilon_\mu(\mathbf{p}, \lambda') = \eta_{\lambda\lambda'}$
- $[a_\lambda(\mathbf{p}), a_{\lambda'}^\dagger(\mathbf{q})] = -\eta_{\lambda\lambda'} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q})$
- $H = \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3} |\mathbf{p}| \left(\sum_{i=1}^3 a_i^\dagger(\mathbf{p}) a_i(\mathbf{p}) - a_0^\dagger(\mathbf{p}) a_0(\mathbf{p}) \right)$
- $|\mathbf{p}, \lambda\rangle = \sqrt{2E_{\mathbf{p}}} a_\lambda^\dagger(\mathbf{p}) |0\rangle$, $E_{\mathbf{p}} = |\mathbf{p}|$

Revision 25 (21.1.2015)

- QED Lagrangian $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\lambda}{2} (\partial_\mu A^\mu)^2 + \bar{\Psi} (i\gamma^\mu D_\mu - m) \Psi$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu = \partial_\mu + ieA_\mu$, $\lambda = 1$ is called Feynman gauge
- photon propagator $\mu \text{---}\text{---}\text{---}\nu \triangleq \frac{-i\eta_{\mu\nu}}{p^2 + i\epsilon}$
- Fermion propagator $\longrightarrow \triangleq \frac{i(\gamma \cdot p + m_0)}{p^2 - m_0^2 + i\epsilon}$
- interaction vertex $\begin{array}{c} \nearrow \\ \searrow \end{array} \text{---}\text{---}\text{---} \triangleq -ie\gamma^\mu$

Revision 26 (26.1.2015)

- 1-loop electron propagator in QED

$$\langle \Omega | T \Psi(x) \bar{\Psi}(y) | \Omega \rangle = x \text{---}\text{---}\text{---} y + x \text{---}\text{---}\text{---} \text{---}\text{---}\text{---} y + \dots \quad (\text{second diagram: self-energy at 1-loop})$$

$$\text{self energy: } -i\Sigma_2(p) = (-ie)^2 \int_{\mathbb{R}^4} \frac{d^4k}{(2\pi)^4} \gamma^\mu \frac{i(k + m_0)}{k^2 - m_0^2 + i\epsilon} \gamma_\mu \frac{-i}{(p-k)^2 - \mu^2 + i\epsilon}$$

Revision 27 (28.1.2015)

- regularised electron mass at 1-loop (order $\mathcal{O}(\alpha^2)$)

$$m - m_0 = \Sigma_2(p = m_0) + \mathcal{O}(\alpha^2) = \frac{\alpha_0}{2\pi} m_0 \int_0^1 dx (2-x) \log \left(\frac{x\Lambda^2}{(1-x)^2 m_0^2 + x\mu^2} \right)$$

gives $m_0 = m_0(m, \Lambda)$ for the bare mass m_0 in terms of the measured physical mass m , and Λ ; plug in everywhere for bare quantities expressions of the form $m_0(m, \Lambda)$, etc., then Λ disappears for all physical quantities such as scattering amplitudes (this process is called renormalisation)

- Photon propagator (in Feynman gauge)

$$\langle \Omega | T A_\mu(x) A_\nu(y) | \Omega \rangle = \int_{\mathbb{R}^4} \frac{d^4q}{(2\pi)^4} e^{-iq(x-y)} \left(\frac{-i}{q^2(1 - \Pi(q^2))} \left(\eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) + \frac{-i}{q^2} \frac{q_\mu q_\nu}{q^2} \right)$$