General Relativity - Mock Exam

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Disclaimer: The amount of work on this sheet might not be equivalent to the actual exam.

Useful formulae:

$$\begin{aligned}
\Gamma^{\alpha}{}_{\mu\nu} &= \frac{1}{2} g^{\alpha\sigma} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right) \\
R_{\mu\nu} &= \partial_\alpha \Gamma^{\alpha}{}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}{}_{\mu\alpha} + \Gamma^{\beta}{}_{\mu\nu}\Gamma^{\alpha}{}_{\beta\alpha} - \Gamma^{\beta}{}_{\mu\alpha}\Gamma^{\alpha}{}_{\beta\nu} \\
R &= g^{\mu\nu} R_{\mu\nu} \\
\mathrm{d}s^2_{\mathrm{FLRW}} &= c^2\,\mathrm{d}t^2 - a^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\Omega^2\right)
\end{aligned}$$

1 Warmup

Answer in short sentences, you may or may not need to use formulae to explain your answers. Just use common sense.

a) By which property is inert mass the same as gravitational mass?

This is due to the fact that

> In an arbitrary gravitational field, no local non-gravitational experiment can distinguish a freely falling, non-rotating system from a uniformly moving system in absence of the gravitational field.

The above quote is *Einstein's Equivalence Principle*, a heuristic guiding principle for the construction of general relativity.

b) Describe one way Newtonian gravity fails to predict empirical results!

Mercury's orbit is not closed. It exhibits a precession of $47''$ per century not included in a description based on Newtonian gravity.

Newtonian gravity also fails to explain gravitational lensing.

c) What is the difference between the idea of a Lorentz transformation and a Galilean transformation?

Galilean transformations treat time- and space-coordinates separately, the idea behind being that every observer agrees on a universal time, $t = t'$, irrespective of the frame of reference. Lorentz transformations transform both time- and space-coordinates equally. The paradigm here that the speed of light is the same for every observer whereas the flow of time depends on the velocity, i.e. the frame of reference. This principle is known as relativity.

d) What is the normalisation condition for a lightlike vector k^ν?

$k^2 = g_{\mu\nu} k^\mu k^\nu = 0.$

e) What is the Ricci-Scalar of Euclidean space in spherical coordinates?

$R = 0$. The Ricci-scalar characterizes the curvature of a topological space. Euclidean space is flat, hence it's Ricci-scalar is zero (in every coordinate system).

f) Why is a global 4-dimensional vector space not sufficient to explain/calculate gravity?

Because vector spaces are linear and can therefore only model linear geometries.

Einstein's breakthrough was to replace Minkowski spacetime with a curved spacetime, where the curvature $R_{\mu\nu}$ was created by and reacted back on energy and momentum $T_{\mu\nu}$. This two-way feedback is the source of *non-linearity* in general relativity and is captured by Einstein's field equations,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \,, \tag{1}$$

where $G_{\mu\nu}$ is the so-called Einstein tensor and $R_{\mu\nu}$ is the Ricci curvature tensor.

g) What do the Christoffel symbols describe?

Christoffel symbols describe how basis vectors change when moving between tangent spaces of different points on a manifold. This allows them to carry out the effects of parallel transport $\frac{\mathrm{d}v^\mu}{\mathrm{d}\lambda} + \Gamma^\mu{}_{\rho\sigma} \frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda} v^\sigma = 0$ on a vector v^μ along a curve parametrized by λ.

h) Why do we have covariant and contravariant quantities?

Because geometric objects can behave differently under Lorentz transformations. Contractians of covariant and contravariant objects are used to construct Lorentz invariant quantities.

i) Why can we transform $g_{\mu\nu}$ always into a diagonal form?

Because in any basis, the metric's components are given by the scalar product of the basis vectors. The metric is thus symmetric

$$g_{\mu\nu} = \boldsymbol{e}_\mu \cdot \boldsymbol{e}_\nu = \boldsymbol{e}_\nu \cdot \boldsymbol{e}_\mu = g_{\nu\mu}, {}^{1} \tag{2}$$

and for any symmetric matrix $\underline{S}$ there exists an orthogonal matrix $\underline{O}$ such that

$$\underline{S} = \underline{O}^T \, \underline{D} \, \underline{O}, \tag{3}$$

where $\underline{D}$ is diagonal.

j) What is a Killing vector?

Killing vectors are used to characterize symmetries in a coordinate independent way. They are the generators of a symmetry in the sense that moving each point on an object the same distance in the direction of the Killing vector field will not distort distances. A vector field $V^\mu(x)$ is a Killing vector field if the Lie derivative of the metric w.r.t this vector field vanishes, i.e. if

$$\mathcal{L}_V g_{\mu\nu} = 0. \tag{4}$$

2 Christoffel symbol transformation properties

a) Show that the Christoffel symbols $\Gamma^\alpha{}_{\mu\nu}$ are not tensors.

A tensor $\underline{T}$ of rank (k, l) is a multilinear map that takes k dual vectors and l vectors and projects them onto a number in $\mathbb{R}$. $\underline{T}$ can be expanded as

$$\underline{T} = T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} \, \partial_{\mu_1} \otimes \dots \otimes \partial_{\mu_k} \otimes \mathrm{d}x^{\nu_1} \otimes \dots \otimes \mathrm{d}x^{\nu_l}. \tag{5}$$

To qualify as a tensor, rather than just an array of numbers, the components of a tensor have to fulfill a certain transformation law. Under the transformation $x^\mu \to x'^\mu$ the components have to change according to

$$T^{\mu_1' \dots \mu_k'}{}_{\nu_1' \dots \nu_l'} = \frac{\partial x^{\mu_1'}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{\mu_k'}}{\partial x^{\mu_k}} \, \frac{\partial x^{\nu_1}}{\partial x^{\nu_1'}} \cdots \frac{\partial x^{\nu_l}}{\partial x^{\nu_l'}} \; T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} \,. \tag{6}$$

[1] This relation for the components of the metric does not hold in non-torsion-free spaces.

The transformation law for the Christoffel symbols follows from the requirements imposed on the covariant derivative. ∇ is set up to perform the functions of the partial derivative ∂, but in a coordinate-independent way. Since we also want it to remain a linear operator obeying the product rule, it can be written as the partial derivative plus a linear transformation, applied as a correction after the partial derivative to make the result covariant. Thus, by requiring that

$$\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma^\nu{}_{\mu\lambda} v^\lambda \tag{7}$$

for some vector v^ν transform as a tensor, we immediately see that $\Gamma^\nu{}_{\mu\lambda}$ cannot transform as a tensor because ∂_μ doesn't. We quantify this assertion be calculating the tranformation $\nabla_\mu v^\nu \to \nabla_{\mu'} v^{\nu'}$, where as in eq. (6),

$$\nabla_{\mu'} v^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \nabla_\mu v^\nu. \tag{8}$$

We can evaluate both sides of eq. (8). For the left, we find

$$\nabla_{\mu'} v^{\nu'} = \partial_{\mu'} v^{\nu'} + \Gamma^{\nu'}{}_{\mu'\lambda'} v^{\lambda'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu v^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} v^\nu \partial_\mu \frac{\partial x^{\nu'}}{\partial x^\nu} + \Gamma^{\nu'}{}_{\mu'\lambda'} \frac{\partial x^{\lambda'}}{\partial x^\lambda} v^\lambda. \tag{9}$$

Meanwhile, the right side yields

$$\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \nabla_\mu v^\nu = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu v^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu{}_{\mu\lambda} v^\lambda. \tag{10}$$

Comparing eqs. (9) and (10), we see that the first term in each is identical. That leaves

$$\Gamma^{\nu'}{}_{\mu'\lambda'} \frac{\partial x^{\lambda'}}{\partial x^\lambda} v^\lambda = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu{}_{\mu\lambda} v^\lambda - \underbrace{\frac{\partial x^\mu}{\partial x^{\mu'}} v^\lambda \partial_\mu \frac{\partial x^{\nu'}}{\partial x^\lambda}}_{\nu \to \lambda}, \tag{11}$$

where we renamed the index ν to λ in the last term. The vector v^ν was arbitrary, so eq. (11) must hold for any v^ν. Hence, we may as well eliminate it on both sides. Lastly, by multiplying with $\frac{\partial x^\lambda}{\partial x^{\lambda'}}$, we can isolate $\Gamma^{\nu'}{}_{\mu'\lambda'}$ to find:

$$\Gamma^{\nu'}{}_{\mu'\lambda'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu{}_{\mu\lambda} - \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda}. \tag{12}$$

Evidently, this does not fit the tensorial transformation behavior of eq. (6); the second term is surplus.

b) Show that $S^\alpha{}_{\mu\nu} = \Gamma^\alpha{}_{\mu\nu} - \Gamma^\alpha{}_{\nu\mu}$ is a tensor.

$$\begin{aligned} S^{\alpha'}{}_{\mu'\nu'} &= \Gamma^{\alpha'}{}_{\mu'\nu'} - \Gamma^{\alpha'}{}_{\nu'\mu'} \\ &= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial x^{\alpha'}}{\partial x^\alpha} \Gamma^\alpha{}_{\mu\nu} - \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial^2 x^{\alpha'}}{\partial x^\mu \partial x^\nu} - \frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\alpha'}}{\partial x^\alpha} \Gamma^\alpha{}_{\nu\mu} + \frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial^2 x^{\alpha'}}{\partial x^\nu \partial x^\mu} \\ &= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial x^{\alpha'}}{\partial x^\alpha} \left(\Gamma^\alpha{}_{\mu\nu} - \Gamma^\alpha{}_{\nu\mu}\right) = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial x^{\alpha'}}{\partial x^\alpha} S^\alpha{}_{\mu\nu} \end{aligned} \tag{13}$$

$S^\alpha{}_{\mu\nu}$ fit's eq. (6)'s transformation law perfectly. Thus, it is indeed a rank $(1, 2)$-tensor.

c) Using the result from part b), prove that in Riemannian geometry $\Gamma^\alpha{}_{\mu\nu} = \Gamma^\alpha{}_{\nu\mu}$.

The tensor $S^\alpha{}_{\mu\nu}$ defined in part b) is just the torsion tensor. Since a Riemannian geometry is, by definition, torsion-free, we have $0 = S^\alpha{}_{\mu\nu} = \Gamma^\alpha{}_{\mu\nu} - \Gamma^\alpha{}_{\nu\mu}$ and thus $\Gamma^\alpha{}_{\mu\nu} = \Gamma^\alpha{}_{\nu\mu}$.

3 Radial infall towards a Kerr Black Hole

The metric of a Kerr black hole is given by the components

$$g_{tt} = -\left(1 - \frac{2m}{r}\right), \qquad g_{t\phi} = -\frac{2mr}{\rho^2} a \sin^2\theta, \qquad g_{rr} = \frac{\rho^2}{\Delta}, \qquad g_{\theta\theta} = \rho^2, \qquad g_{\phi\phi} = \frac{\Sigma^2}{\rho^2}\sin^2\theta.$$

with the functions

$$\Delta = r^2 - 2mr + a^2, \qquad \rho^2 = r^2 + a^2\cos^2\theta, \qquad \Sigma^2 = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta.$$

Set up the 'Lagrangian' for a test particle falling from $r = \infty$ to $r = 0$

a) **radially in the equatorial plane.**

The action can be written in terms of the line element $\mathrm{d}s$ as

$$S = -mc\int \mathrm{d}s = -mc\int \frac{\mathrm{d}s}{\mathrm{d}\lambda}\,\mathrm{d}\lambda = -mc\int \sqrt{g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\,\mathrm{d}\lambda, \tag{14}$$

where λ is an affine parameter of the geodesic over which we integrate.[2] The name affine parameter stems from its connection to the proper time τ via an affine transformation $\tau \to \lambda = a\tau + b$ for some constant a and b. $\mathrm{d}\lambda = a\,\mathrm{d}\tau$ is just a scaled proper time, so we may identify our Lagrangian with

$$L = -mc\sqrt{g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}. \tag{15}$$

The equations of motion are unaffected a rescaling of the Lagrangian, so we drop the unwelcome prefactor $-mc$ and also the square root. Defining $L' = L^2$, we see that the latter's effect on the equations of motion amounts to

$$0 = \frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L'}{\partial \dot{x}^\mu} - \frac{\partial L'}{\partial x^\mu} = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(2L\frac{\partial L}{\partial \dot{x}^\mu}\right) - 2L\frac{\partial L}{\partial x^\mu}, \tag{16}$$

where we can simply divide by $2L$ since $L = \frac{\mathrm{d}s}{\mathrm{d}\lambda} = \frac{c\,\mathrm{d}\tau}{a\,\mathrm{d}\tau} = \frac{c}{a}$ is just a constant.

With these simplifications and denoting $\frac{\partial x^\mu}{\partial \tau} = \dot{x}^\mu$ the Lagrangian reads

$$\begin{aligned} L &= g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = g_{tt}\,\dot{t}^2 + g_{t\phi}\,\dot{t}\,\dot{\phi} + g_{rr}\,\dot{r}^2 + g_{\theta\theta}\,\dot{\theta}^2 + g_{\phi\phi}\,\dot{\phi}^2 \\ &= -\left(1 - \frac{2m}{r}\right)\dot{t}^2 - \frac{2mr}{\rho^2}a\sin^2\theta\,\dot{t}\,\dot{\phi} + \frac{\rho^2}{\Delta}\,\dot{r}^2 + \rho^2\,\dot{\theta}^2 + \frac{\Sigma^2}{\rho^2}\sin^2\theta\,\dot{\phi}^2. \end{aligned} \tag{17}$$

However, since we are tasked with considering the specific case of a radially inbound test particle moving in the equatorial plane, we may set all time derivatives of angles to zero, $\dot{\theta} = \dot{\phi} = 0$ and $\theta = \pi/2$. Thus,

$$L = -\left(1 - \frac{2m}{r}\right)\dot{t}^2 + \frac{r^2 + a^2\overbrace{\cos^2\theta}^{0}}{r^2 - 2mr + a^2}\,\dot{r}^2 = -\left(1 - \frac{2m}{r}\right)\dot{t}^2 + \frac{\dot{r}^2}{1 - \frac{2m}{r} + \frac{a^2}{r^2}}. \tag{18}$$

b) **along the polar axis.**

Travelling along the polar axis from $r = \infty$ to $r = 0$ still implies $\dot{\theta} = \dot{\phi} = 0$. However, now $\theta = 0$ so that $\cos^2\theta = 1$. We therefore have to modify the result of eq. (18) only slightly,

$$L = -\left(1 - \frac{2m}{r}\right)\dot{t}^2 + \frac{r^2 + a^2}{r^2 - 2mr + a^2}\,\dot{r}^2 = -\left(1 - \frac{2m}{r}\right)\dot{t}^2 + \frac{\dot{r}^2}{1 - \frac{2mr}{r^2 + a^2}} \tag{19}$$

[2] A geodesic is the curved-space generalization of the notion of a "straight line" in Euclidean space.

c) Derive the equations of motion for r in both cases, assuming that the particle is at rest at $r = \infty$.

The Euler-Lagrange equation for r reads

$$0 \stackrel{!}{=} \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r}. \tag{20}$$

For the scenario described in part a), we get

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{2\dot{r}}{1-\frac{2m}{r}+\frac{a^2}{r^2}}\right) + \frac{2m}{r^2}\dot{t}^2 + \frac{\dot{r}^2}{\left(1-\frac{2m}{r}+\frac{a^2}{r^2}\right)^2}\left(\frac{2m}{r^2}-\frac{2a^2}{r^3}\right)\\
&= \frac{2\ddot{r}}{1-\frac{2m}{r}+\frac{a^2}{r^2}} - \frac{2\dot{r}}{\left(1-\frac{2m}{r}+\frac{a^2}{r^2}\right)^2}\left(\frac{2m}{r^2}-\frac{2a^2}{r^3}\right)\dot{r} + \frac{2m}{r^2}\dot{t}^2 + \frac{\dot{r}^2}{\left(1-\frac{2m}{r}+\frac{a^2}{r^2}\right)^2}\left(\frac{2m}{r^2}-\frac{2a^2}{r^3}\right)\\
\Rightarrow \quad 0 &= \ddot{r} - \frac{\left(\frac{m}{r^2}-\frac{a^2}{r^3}\right)\dot{r}^2}{1-\frac{2m}{r}+\frac{a^2}{r^2}} + \frac{m}{r^2}\dot{t}^2\left(1-\frac{2m}{r}+\frac{a^2}{r^2}\right) = \ddot{r} - \frac{(mr-a^2)\,\dot{r}^2}{r^3-2mr+a^2r} + \frac{m}{r^2}\dot{t}^2\left(1-\frac{2m}{r}+\frac{a^2}{r^2}\right),
\end{aligned} \tag{21}$$

whereas the one from part b) yields

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{2\dot{r}}{1-\frac{2mr}{r^2+a^2}}\right) + \frac{2m}{r^2}\dot{t}^2 + \frac{\dot{r}^2}{\left(1-\frac{2mr}{r^2+a^2}\right)^2}\left(-\frac{2m}{r^2+a^2}+\frac{2mr}{(r^2+a^2)^2}2r\right)\\
&= \frac{2\ddot{r}}{1-\frac{2mr}{r^2+a^2}} - \frac{2\dot{r}}{\left(1-\frac{2mr}{r^2+a^2}\right)^2}\left(-\frac{2m}{r^2+a^2}+\frac{2mr}{(r^2+a^2)^2}2r\right)\dot{r}\\
&\quad + \frac{2m}{r^2}\dot{t}^2 + \frac{\dot{r}^2}{\left(1-\frac{2mr}{r^2+a^2}\right)^2}\left(-\frac{2m}{r^2+a^2}+\frac{2mr}{(r^2+a^2)^2}2r\right)\\
\Rightarrow \quad 0 &= \ddot{r} - \frac{\left(\frac{2mr^2}{(r^2+a^2)^2}-\frac{m}{r^2+a^2}\right)\dot{r}^2}{1-\frac{2mr}{r^2+a^2}} + \frac{m}{r^2}\dot{t}^2\left(1-\frac{2mr}{r^2+a^2}\right)\\
&= \ddot{r} - \frac{m(r^2-a^2)\dot{r}^2}{(r^2+a^2)(r^2+a^2-2mr)} + \frac{m}{r^2}\dot{t}^2\left(1-\frac{2mr}{r^2+a^2}\right).
\end{aligned} \tag{22}$$

d) Why do they not agree? Do they, if $a = 0$?

As expected, the equations of motion for a particle inbound radially in the equatorial plane and along the polar axis differ. The reason is that the metric is not isotropic: space looks differently when viewed from the origin $r = 0$ depending on the direction.

If however, we set $a = 0$ we would expect the equations to agree since all angle dependencies in our metric come with a factor of a. Indeed, we find for both cases,

$$0 = \ddot{r} + \frac{m\dot{r}^2}{2mr - r^2} + \frac{m}{r^2}\dot{t}^2\left(1-\frac{2m}{r}\right). \tag{23}$$

4 Metric comparison

Given the line element

$$\mathrm{d}s^2 = \mathrm{d}u^2 + \cosh^2 u\,\mathrm{d}v^2$$

we notice its similarity with two-dimensional radial coordinates

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\phi^2.$$

So let's call u and r the radii in Lobatchevski space and Euclidean space, respectively.

a) What are the circumferences of a circle in both spaces?

Geometrically, a circle is defined as the set of all points that are at a given distance r, the radius, from a given point c, the center. On a smooth manifold, this set of points forms a closed path γ. The length of this path can be calculated from the line element via

$$I[\gamma] = \int_\gamma \mathrm{d}s = \int_\gamma \sqrt{g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu}. \tag{24}$$

We assume that in both Euclidean and Lobatchevski space, the radii take values in $r \in (0,\infty) \ni u$ and the angles $\phi \in (0, 2\pi) \ni v$. For the case of radial coordinates, inserting the above line element then gives

$$I_\mathrm{E} = \int_\gamma \sqrt{\mathrm{d}r^2 + r^2\,\mathrm{d}\phi^2} = \int_0^{2\pi} \sqrt{\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 + r^2}\;\mathrm{d}\phi = \int_0^{2\pi} r\;\mathrm{d}\phi = 2\pi r, \tag{25}$$

where $\mathrm{d}r/\,\mathrm{d}\phi = 0$, because we take r to be the radius of our circle which is not supposed to change along the length of it. Using the same argument for Lobatchevski coordinates, we find that in this space, the circumference of a circle is given by

$$I_\mathrm{L} = \int_\gamma \sqrt{\mathrm{d}u^2 + \cosh^2 u\,\mathrm{d}\phi^2} = \int_0^{2\pi} \sqrt{\left(\frac{\mathrm{d}u}{\mathrm{d}v}\right)^2 + \cosh^2 u}\;\mathrm{d}\phi = \int_0^{2\pi} \cosh u\;\mathrm{d}\phi = 2\pi\cosh u. \tag{26}$$

b) What are the surfaces of circles in both spaces?

If metric is symmetric, the line element can be rewritten as $\mathrm{d}s = \prod_\mu \sqrt{g_{\mu\mu}}\,\mathrm{d}x^\mu$. For spherical coordinates, this gives

$$\mathrm{d}s = \sqrt{g_{rr}g_{\phi\phi}}\,\mathrm{d}r\,\mathrm{d}\phi = r\,\mathrm{d}r\,\mathrm{d}\phi, \tag{27}$$

and for Lobatchevski coordinates

$$\mathrm{d}s = \sqrt{g_{uu}g_{vv}}\,\mathrm{d}u\,\mathrm{d}v = \cosh u\,\mathrm{d}u\,\mathrm{d}v. \tag{28}$$

Integrating this measure over all angles from 0 to r and u, respectively, yields

$$A_\mathrm{E} = \int_0^{2\pi}\int_0^r r'\,\mathrm{d}r'\,\mathrm{d}\phi' = 2\pi\,\frac{1}{2}r'^2\Big|_0^r = \pi r^2 \tag{29}$$

and

$$A_\mathrm{L} = \int_0^{2\pi}\int_0^u \cosh u'\,\mathrm{d}u'\,\mathrm{d}v' = 2\pi\,\sinh u'\big|_0^u = 2\pi\sinh u \tag{30}$$

c) How do you explain the differences? How do you interpret them?

Euclidean space is flat, Lobatchevski space is curved. Curvature has an effect on lengths and areas. In this case, since Lobatchevski space is hyperbolic, i.e. has a constant negative curvature (think wormhole or a cylinder compressed in the middle), both length and area of a circle should increase much more rapidly with radius, than in Euclidean space. Indeed, $\cosh u = \frac{1}{2}(e^{-u} + e^u)$ grows much faster than r.

d) Can you bring both line elements into Cartesian form, i.e. $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$?

Two-dimensional Cartesian coordinates (x, y) are given in terms of polar coordinates (r, ϕ) by

$$\begin{pmatrix} x(r,\phi) \\ y(r,\phi) \end{pmatrix} = \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix}. \tag{31}$$

This relation can be inverted:

$$\begin{pmatrix} r(x,y) \\ \phi(x,y) \end{pmatrix} = \begin{pmatrix} \sqrt{x^2+y^2} \\ \arctan\left(\frac{x}{y}\right) \end{pmatrix}. \tag{32}$$

Therefore, $\mathrm{d}r = \mathrm{d}r(x,y) = \frac{x\,\mathrm{d}x}{\sqrt{x^2+y^2}} + \frac{y\,\mathrm{d}y}{\sqrt{x^2+y^2}}$ and $\mathrm{d}\phi = \mathrm{d}\phi(x,y) = \frac{y\,\mathrm{d}x}{x^2+y^2} - \frac{x\,\mathrm{d}y}{x^2+y^2}$. Inserting these relations into the line element $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\phi^2$, we recover the Cartesian line element:

$$\begin{aligned}\mathrm{d}s^2 &= \left(\frac{x\,\mathrm{d}x}{\sqrt{x^2+y^2}} + \frac{y\,\mathrm{d}y}{\sqrt{x^2+y^2}}\right)^2 + (x^2+y^2)\left(\frac{y\,\mathrm{d}x}{x^2+y^2} - \frac{x\,\mathrm{d}y}{x^2+y^2}\right)^2 \\ &= \frac{x^2\,\mathrm{d}x^2}{x^2+y^2} + \frac{2xy\,\mathrm{d}x\,\mathrm{d}y}{x^2+y^2} + \frac{y^2\,\mathrm{d}y^2}{x^2+y^2} + \frac{y^2\,\mathrm{d}x^2}{x^2+y^2} - \frac{2xy\,\mathrm{d}x\,\mathrm{d}y}{x^2+y^2} + \frac{x^2\,\mathrm{d}y^2}{x^2+y^2} \\ &= \frac{1}{x^2+y^2}\left(x^2\,\mathrm{d}x^2 + y^2\,\mathrm{d}y^2 + y^2\,\mathrm{d}x^2 + x^2\,\mathrm{d}y^2\right) = \mathrm{d}x^2 + \mathrm{d}y^2\end{aligned} \tag{33}$$

Since radial coordinates are simply an adapted set of coordinates to describe spherically symmetric systems *in flat Euclidean space*, it should come as no surprise, that we were able to transform the line element back into Cartesian coordinates.

Lobatchevski space, however, is hyperbolic. It has curvature, regardless of the coordinate set used to describe it. Thus, it is impossible to bring its line element into Cartesian form.

5 Friedmann universe

a) What are the two big assumptions in FLRW-cosmology and how do they manifest in the line element $\mathrm{d}s^2$?

A Friedmann-Lemaître-Robertson-Walker cosmology assumes the universe is isotropic and homogeneous. These assumptions are sufficient to derive the general form of the metric (i.e. Einstein's field equations are not required for this; they only serve to determine the scale factor $a^2(t)$). The line element reads

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right]. \tag{34}$$

For every FLRW-metric, there exists a set of coordinates such that the spatial part of the metric is time-independent.

b) The line element seems to be singular at a certain radius r; does the Universe become unphysical at this radius?

Indeed, at $r = 1/\sqrt{k}$, the g_{rr}-component of the metric diverges. However, this is a coordinate singularity and not physically rooted. The Ricci-scalar of eq. (34) is

$$R = -\frac{6}{a^2}(a\ddot{a} + \dot{a}^2 + k). \tag{35}$$

It is completely independent of r, θ, and ϕ, i.e. spatially homogeneous, and in particular, does not diverge for $r = 1/\sqrt{k}$.

c) Write down the radial equation of motion of a particle in a FLRW-cosmology. Does Newton's first law apply?

Following the argument made in problem 3, we take as our Lagrangian again the square of the line element differentiated with respect to the proper time τ, i.e.

$$L = g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = c^2\dot{t}^2 - a^2(t)\left[\frac{\dot{r}^2}{1-kr^2} + r^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right)\right]. \tag{36}$$

Then the Euler-Lagrange equation for the radial coordinate r reads

$$
\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(-a^2(t)\frac{2\dot{r}}{1-kr^2}\right) + a^2(t)\left(\frac{2kr\dot{r^2}}{(1-kr^2)^2} + 2r\left(\dot{\theta^2} + \sin^2\theta\,\dot{\phi^2}\right)\right) \\
&= -2a(t)\dot{a}(t)\dot{t}\frac{2\dot{r}}{1-kr^2} - a^2(t)\frac{2\ddot{r}}{1-kr^2} - a^2(t)\frac{4kr\dot{r^2}}{(1-kr^2)^2} \\
&\quad + a^2(t)\left(\frac{2kr\dot{r^2}}{(1-kr^2)^2} + 2r\left(\dot{\theta^2} + \sin^2\theta\,\dot{\phi^2}\right)\right) \\
\Rightarrow \qquad 0 &= \ddot{r} + 2\frac{\dot{a}(t)}{a(t)}\dot{t}\dot{r} + \frac{kr\dot{r}^2}{1-kr^2} - r(1-kr^2)\left(\dot{\theta^2} + \sin^2\theta\,\dot{\phi^2}\right).
\end{aligned} \tag{37}
$$

The question is not entirely unambiguous. It could be that by 'radial equation of motion' not the Euler-Lagrange of the coordinate r is meant but instead (as in problem 3) the equation of motion of a particle moving only radially. In that case, eq. (37) reduces to

$$
0 = \ddot{r} + 2\frac{\dot{a}(t)}{a(t)}\dot{t}\dot{r} + \frac{kr\dot{r}^2}{1-kr^2} \tag{38}
$$

Newton's first law states that when viewed in an inertial reference frame, an object either remains at rest or continues to move at a constant velocity without change of direction, unless acted upon by an external force. Considering the forceless case, $\ddot{r} = 0$, we find

$$
0 = 2H(t)\dot{t} + \frac{kr\dot{r}}{1-kr^2} \quad \Rightarrow \quad \frac{\dot{r}}{\dot{t}} = \frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}t} = -2H(t)\frac{1-kr^2}{kr}, \tag{39}
$$

where $H(t) = \dot{a}(t)/a(t)$ is the Hubble parameter. Equation (39) clearly shows that the velocity changes over time even if no forces act upon the test particle. This is clearly in violation of Newton's first law.

d) The covariant Ricci-tensor in this metric is

$$
\underline{R} = \begin{pmatrix} -3\frac{\ddot{a}}{a} & 0 & 0 & 0 \\ 0 & \frac{a\ddot{a}+2\dot{a}^2+2k}{1-kr^2} & 0 & 0 \\ 0 & 0 & r^2(a\ddot{a}+2\dot{a}^2+2k) & 0 \\ 0 & 0 & 0 & r^2(a\ddot{a}+2\dot{a}^2+2k)\sin^2\theta \end{pmatrix}.
$$

Find the Ricci scalar! Why is it finite even if $k = 0$?

To find the Ricci-scalar given in eq. (35), we contract the Ricci-tensor $R = g^{\mu\nu}R_{\mu\nu}$, where $g^{\mu\nu}$ is just $\frac{1}{g_{\mu\nu}}$ due to the reciprocity of the metric, $g^{\mu\nu}g_{\nu\lambda} = \delta^{\mu}{}_{\lambda}$ and the FLRW-metric being symmetric. Thus,

$$
\begin{aligned}
R &= -3\frac{\ddot{a}}{a} - \frac{1-kr^2}{a^2}\frac{a\ddot{a}+2\dot{a}^2+2k}{1-kr^2} - \frac{1}{a^2r^2}r^2(a\ddot{a}+2\dot{a}^2+2k) \\
&\quad - \frac{1}{a^2r^2\sin^2\theta}r^2(a\ddot{a}+2\dot{a}^2+2k)\sin^2\theta \\
&= -3\frac{\ddot{a}}{a} - 3\frac{a\ddot{a}+2\dot{a}^2+2k}{a^2} = -\frac{6}{a^2}(a\ddot{a}+\dot{a}^2+k).
\end{aligned} \tag{40}
$$

For $k = 0$, we have a curvature of $R = -\frac{6}{a^2}(a\ddot{a}+\dot{a}^2)$, which is indeed finite. It is again not clearly stated, but we assume that the question is actually aimed at an explanation of why R is not zero. This is because $k = 0$ only eliminates spatial curvature. The Ricci-scalar, however, is a measure of the curvature of space*time*. Spacetime as a whole is, of course, still bent due to the presence of the scale factor $a(t)$.