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Quantum Field Theory II – Final Exam by Tillmann Plehn

Problem 1: One-loop calculations in QED

1. QED with massless fermion field $\Psi(x)$ and massless gauge field $A_\mu(x)$

$$\mathcal{L} = -\frac{1}{4} F_{0,\mu\nu} F_0^{\mu\nu} + i \bar{\Psi}_0 \not{D}_0 \Psi_0 , \qquad D_0^\mu = \partial^\mu + i e_0 A_0^\mu$$

We use the Feynman rules to derive the one-loop corrections to the fermion propagator $i\mathcal{M}[f(p) \to f(p)]$ of a fermion f with mom. p. To apply renormalized perturbation theory, we split the bare Lagrangian into renormalized quantities, e.g. $\Psi = Z_\Psi^{-\frac{1}{2}} \Psi_0$ and $A^\mu = Z_A^{-\frac{1}{2}} A_0^\mu$, and counter-terms as follows

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \bar{\Psi} \not{\partial} \Psi - e A_\mu \bar{\Psi} \gamma^\mu \Psi - \frac{1}{4} \delta_A F_{\mu\nu} F^{\mu\nu} + i \delta_\Psi \bar{\Psi} \not{\partial} \Psi - \delta_e A_\mu \bar{\Psi} \gamma^\mu \Psi$$

where $\delta_A = Z_A - 1$, $\delta_\Psi = Z_\Psi - 1$, and $\delta_e = e_0 Z_A^{\frac{1}{2}} Z_\Psi - e$.

Integrating by parts the photonic counterterm yields

$$S_\gamma = \int d^4x \left(-\frac{1}{4} \delta_A F_{\mu\nu} F^{\mu\nu} \right) = -\frac{1}{4} \delta_A \int d^4x \big(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu$$
$$- \partial_\nu A_\mu \partial^\mu A^\nu + \partial_\nu A_\mu \partial^\nu A^\mu \big)$$

$$= -\frac{1}{2} \delta_A \int d^4x \left(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu \right)$$

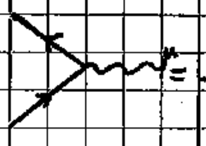
$$= -\frac{1}{2} \delta_A \int d^4x \left(- A^\nu \partial_\mu \partial^\mu A^\nu + A_\nu \partial_\mu \partial^\nu A^\mu \right)$$

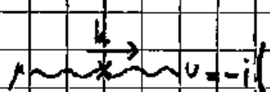
$$= -\frac{1}{2} \delta_A \int d^4x \, A_\nu \left(-\partial^2 g^{\mu\nu} + \partial^\mu \partial^\nu \right) A_\mu .$$

So our complete set of Feynman rules yields


$= \frac{-i g_{\mu\nu}}{k^2}$


$= \frac{i}{\not{p}}$

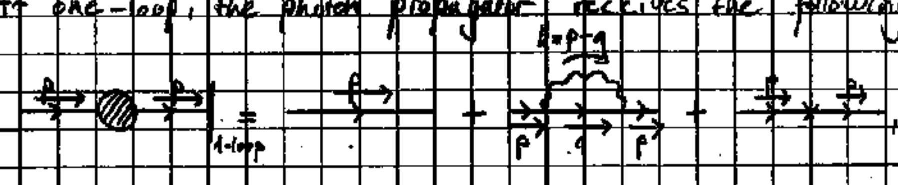

$= -ie\gamma^\mu$


$= -i(k^2 g_{\mu\nu} - k_\mu k_\nu)\delta_A$


$= i(\not{p}\,\delta_\Psi - \delta_m)$ ← needs to be added even in massless theories

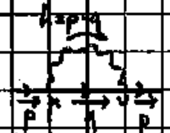

$= -ie\gamma^\mu$

At one-loop, the photon propagator receives the following contributions



of which only the latter two are corrections. We compute

$$= i(\not p\,\delta_\psi - \delta_m)$$



$$= \bar u(p)(-ie\gamma^\nu)\int\frac{d^dq}{(2\pi)^d}\frac{i\not q}{q^2}\frac{-ig_{\mu\nu}}{k^2}(-ie\gamma^\mu)u(p)$$

$$-i\mathcal{M}(f(p)\to f(p)) = -e^2\,\bar u(p)\int\frac{d^dq}{(2\pi)^d}\frac{\gamma_\mu\not q\gamma^\mu}{q^2(p-q)^2}u(p),\quad \gamma_\mu\not q\gamma^\mu = \gamma_\mu(-\gamma^\mu\not q+2q^\mu) = -2\not q$$

$$= -e^2\,\bar u(p)\frac{\Gamma(1+1)}{\Gamma(1)\Gamma(1)}\int_0^1 dx\int\frac{d^dq}{(2\pi)^d}\frac{-2\not q}{(\underbrace{x(p-q)^2+(1-x)q^2}_{xp^2-2xpq+xq^2+q^2-xq^2})^2}$$

$$= 2e^2\,\bar u(p)\int_0^1 dx\int\frac{d^dq}{(2\pi)^d}\frac{\not q}{((\underbrace{q-xp}_{l})^2+x(1-x)p^2)^2}u(p)$$

$$= 8\pi\alpha_{em}\,\bar u(p)\int_0^1 dx\int\frac{d^dl}{(2\pi)^d}\frac{\overbrace{\not l}^{0,\ \text{odd}}+x\not p}{(l^2+x(1-x)p^2)^2}u(p)$$

$$= 8\pi\alpha_{em}\,\bar u(p)\not p F(p^2)u(p),\qquad (\not p u(p)=0)$$

where we calculate $F(p^2)$ using the subst. $z=\frac{\Delta}{l_E^2-\Delta}$, $l_E=\Delta^{\frac12}(\frac1z-1)^{\frac12}$,

$$F(p^2) = i\int_0^1 dx\int\frac{d^dl_E}{(2\pi)^d}\frac{x}{(l_E^2-\underbrace{x(1-x)p^2}_{\Delta})^2} = i\int_0^1 dx\,x\int\frac{d\Omega_d}{(2\pi)^d}\int_0^\infty dl_E\frac{l_E^{d-1}}{(l_E^2-\Delta)^2}$$

$$= i\int_0^1 dx\,x\,\frac{2\pi^{\frac d2}}{(2\pi)^d\Gamma(\frac d2)}\int_1^0 dz\frac{(l_E-\Delta)^2}{-2\Delta l_E}\frac{l_E^{d-1}}{(l_E-\Delta)^2}$$

$$= \frac{1}{(4\pi)^{\frac d2}\Gamma(\frac d2)}\int_0^1 dx\,x\int_0^1 dz\,\Delta^{\frac d2-2}z^{1-\frac d2}(1-z)^{\frac d2-1}$$

$$= \frac{1}{(4\pi)^{\frac d2}\Gamma(\frac d2)}\frac{\Gamma(2-\frac d2)\Gamma(\frac d2)}{\Gamma(2-\frac d2+\frac d2)}\int_0^1 dx\,x\,\Delta^{\frac d2-2} = \frac{\Gamma(\frac\epsilon2)}{(4\pi)^{d/2}}\int_0^1 dx\,x\,(x(1-x)p^2)^{\frac d2-2}$$

To bring out the divergence, we expand the integrand.

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$$\int_0^1 dx\, x\,(x(1-x)p^2)^{\frac{\epsilon}{2}} = \int_0^1 dx\, x\left(1+\frac{\epsilon}{2}\ln(x(1-x)p^2)+\sigma(\epsilon^2)\right)$$

$$=\frac{1}{2}+\frac{\epsilon}{2}\int_0^1 dx\, x\,\ln(x(1-x)p^2)+\sigma(\epsilon^2).$$

We can also expand the prefactors $\Gamma(\frac{\epsilon}{2})=\frac{2}{\epsilon}-\gamma+\sigma(\epsilon)$ and

$$\frac{1}{(4\pi)^{d/2}}=\frac{1}{(4\pi)^{2-\frac{\epsilon}{2}}}=\frac{(4\pi)^{\frac{\epsilon}{2}}}{16\pi^2}=\frac{1}{16\pi^2}\left(1+\frac{\epsilon}{2}\ln(4\pi)+\sigma(\epsilon^2)\right).$$

Assembling everything, we get

$$F(p^2)=\frac{i}{16\pi^2}\left(1+\frac{\epsilon}{2}\ln(4\pi)+\sigma(\epsilon^2)\right)\left(\frac{2}{\epsilon}-\gamma+\sigma(\epsilon)\right)\left(\frac{1}{2}+\frac{\epsilon}{2}\int_0^1 dx\, x\ln(x(1-x)p^2)+\sigma(\epsilon^2)\right)$$

$$=\frac{i}{32\pi^2}\left(\frac{2}{\epsilon}-\gamma+\ln(4\pi)+\int_0^1 dx\, x\ln(x(1-x)p^2)\right)+\sigma(\epsilon)$$

If we choose our renormalization conditions as

$$\Sigma(\not p)|_{\not p=\mu}=0 \qquad \text{and} \qquad \frac{d}{d\not p}\Sigma(\not p)|_{\not p=\mu}=0,$$

we obtain a counterterm δ_ψ of

$$\frac{d}{d\not p}\Sigma(\not p)|_{\not p=\mu}=i\delta_\psi+\frac{d}{d\not p}\left[2e^2\frac{i}{32\pi^2}\left(\frac{2}{\epsilon}-\gamma+\ln(4\pi)+\int_0^1 dx\, x\ln(x(1-x)\not p^2)\right)\right]\Bigg|_{\not p=\mu}$$

$$=i\delta_\psi+\frac{ie^2}{16\pi^2}\int_0^1 dx\,\frac{2x(1-x)\not p}{x(1-x)\not p^2}\Bigg|_{\not p=\mu}=i\delta_\psi+\frac{ie^2}{8\pi^2\mu}=0 \implies \delta_\psi=-\frac{e^2}{8\pi^2\mu}$$

and for δ_m

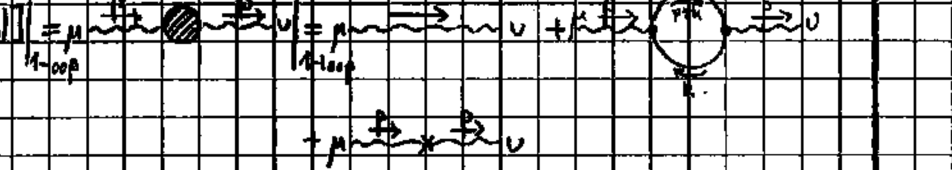
$$\Sigma(\not p)|_{\not p=\mu}=i(\not p\,\delta_\psi-\delta_m)+\frac{ie^2}{16\pi^2}\left(\frac{2}{\epsilon}-\gamma+\ln(4\pi)+\int_0^1 dx\, x\ln(x(1-x)\not p^2)\right)\Bigg|_{\not p=\mu}$$

$$=-\frac{ie^2}{8\pi^2}-\delta_m+\frac{ie^2}{16\pi^2}\left(\frac{2}{\epsilon}-\gamma+\ln(4\pi)+\int_0^1 dx\, x\ln(x(1-x)\mu^2)\right)=0$$

$$\implies \delta_m=\frac{e^2}{16\pi^2}\left(\frac{2}{\epsilon}-\gamma+\ln(4\pi)+\int_0^1 dx\, x\ln(x(1-x)\mu^2)-2\right).$$

2. We calculate the one-loop corrections to the photon propagator, again using the Feynman rules of renormalized perturbation theory.

$-iM[\gamma(p)\to\gamma(p)]\Big|_{1\text{-loop}} =$



The first correction gives

$$= \epsilon^*_\mu(p)\left(-ie(\gamma^\mu)^A{}_B\right)\int\frac{d^dp}{(2\pi)^d}\frac{i(\not p+\not k)^B{}_C}{(p+k)^2}\frac{i(\not k)^C{}_D}{k^2}\left(-ie(\gamma^\nu)^D{}_A\right)\epsilon_\nu(p)$$

$$= -e^2\epsilon^*_\mu(p)\int\frac{d^dk}{(2\pi)^d}\frac{Tr(\gamma^\mu(\not p+\not k)\gamma^\nu\not k)}{k^2(p+k)^2}\epsilon_\nu(p)$$

$$Tr(\gamma^\mu(\not p+\not k)\gamma^\nu\not k) = p_\rho k_\sigma Tr(\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma) + Tr(\gamma^\mu\not k k_\rho(2\eta^{\rho\nu}-\gamma^\rho\gamma^\nu))$$

$$= p_\rho k_\sigma 4(\eta^{\mu\rho}\eta^{\nu\sigma}-\eta^{\mu\nu}\eta^{\rho\sigma}+\eta^{\mu\sigma}\eta^{\rho\nu}) + 2Tr(\gamma^\mu\not k k^\nu) - k^2 Tr(\gamma^\mu\gamma^\nu)$$

$$= 4(p^\mu k^\nu - p^\sigma k_\sigma\eta^{\mu\nu} + p^\nu k^\mu) + 2k_\rho k^\nu Tr(\gamma^\mu\gamma^\rho) - k^2 4\eta^{\mu\nu}$$

$$= 4(p^\mu k^\nu - p\cdot k\,\eta^{\mu\nu} + p^\nu k^\mu) + 8k^\mu k^\nu - 4k^2\eta^{\mu\nu}$$

Using a Feynman parametrization to simplify the denominator

$$iM[\gamma(p)\to\gamma(p)] = -e^2\epsilon^*_\mu(p)\int\frac{d^dk}{(2\pi)^d}\int_0^1 dx\,\frac{Tr(\gamma^\mu(\not p+\not k)\gamma^\nu\not k)}{[x(p+k)^2+(1-x)k^2]^2}\epsilon_\nu(p)$$

$$x(p+k)^2+(1-x)k^2 = xp^2+2xpk+xk^2+k^2-xk^2$$

$$= (k+xp)^2 + x(1-x)p^2 =: L^2 + x(1-x)p^2 = L^2-\Delta$$

where $L = k+xp$, $dL = dk$.

$$iM = -e^2\epsilon^*_\mu(p)\,Tr(\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma)\int_0^1 dx\int\frac{d^dL}{(2\pi)^d}\frac{\overbrace{(L_\rho+(1-x)p_\rho)(L_\sigma-xp_\sigma)}^{L_\rho L_\sigma - x(1-x)p_\rho p_\sigma,\ \text{other terms odd}}}{(L^2-\Delta)^2}\epsilon_\nu(p)$$

$$= -e^2\epsilon^*_\mu(p)\,Tr(\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma)\int_0^1 dx\left[\frac{-i\Gamma(1-\frac{d}{2})}{(4\pi)^{d/2}}\frac{\eta_{\rho\sigma}}{2}\Delta^{\frac{d}{2}-1} - x(1-x)p_\rho p_\sigma\frac{i\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}}\Delta^{\frac{d}{2}-2}\right]\epsilon_\nu(p),$$

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where we used A.44 and A.46 from Peskin & Schroeder to perform the l-integration. We simplify

$$i\mathcal{M} = \frac{ie^2\epsilon^*_\mu}{(4\pi)^{d/2}}\Gamma\left(2-\frac{d}{2}\right)\mathrm{Tr}(\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma)\int_0^1 dx\left(\frac{\eta_{\rho\sigma}}{2-d}(-x(1-x)p^2)^{\frac{d}{2}-1}+x(1-x)p_\rho p_\sigma(-x(1-x)p^2)^{\frac{d}{2}-2}\right)\epsilon_\nu(p)$$

$$= \frac{ie^2}{(4\pi)^{d/2}}\epsilon^*_\mu(p)\Gamma\left(\frac{\epsilon}{2}\right)\mathrm{Tr}(\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma)\left(\frac{p^2\eta_{\rho\sigma}}{2-d}+p_\rho p_\sigma\right)(-p^2)^{\frac{d}{2}-2}\int_0^1 dx\, 2[x(1-x)]^{\frac{d}{2}-1}\epsilon_\nu(p)$$

$$= \frac{ie^2}{(4\pi)^{d/2}}\epsilon^*_\mu(p)\Gamma\left(\frac{\epsilon}{2}\right)\mathrm{Tr}(\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma)(-p^2)^{-\frac{\epsilon}{2}}\left(\frac{-p^2}{2-d}\eta_{\rho\sigma}+p_\rho p_\sigma\right)\frac{\Gamma(\frac{d}{2})\Gamma(\frac{d}{2})}{\Gamma(d)}\epsilon_\nu(p)$$

$$= \frac{ie^2}{(4\pi)^{d/2}}\epsilon^*_\mu(p)\frac{\Gamma(\frac{\epsilon}{2})\Gamma(\frac{d}{2})^2}{\Gamma(d)}(-p^2)^{-\frac{\epsilon}{2}}4(\eta^{\mu\rho}\eta^{\nu\sigma}-\eta^{\mu\nu}\eta^{\rho\sigma}+\eta^{\mu\sigma}\eta^{\rho\nu})\left(\frac{-p^2}{2-d}\eta_{\rho\sigma}+p_\rho p_\sigma\right)\epsilon_\nu(p)$$

$$= \frac{4ie^2}{(4\pi)^{d/2}}\epsilon^*_\mu(p)\frac{\Gamma(\frac{\epsilon}{2})\Gamma(\frac{d}{2})^2}{\Gamma(d)}(-p^2)^{-\frac{\epsilon}{2}}\left(\frac{-p^2}{2-d}\left(\eta^{\mu\nu}-\eta^{\mu\nu}\underbrace{\eta^\rho{}_\rho}_{d}+\eta^{\mu\nu}\right)+\left(p^\mu p^\nu-\eta^{\mu\nu}p^2+p^\mu p^\nu\right)\right)\epsilon_\nu(p)$$

$$= \frac{4ie^2}{(4\pi)^{d/2}}\epsilon^*_\mu(p)\frac{\Gamma(\frac{\epsilon}{2})\Gamma(\frac{d}{2})^2}{\Gamma(d)}(-p^2)^{-\frac{\epsilon}{2}}\left(\frac{-2\eta^{\mu\nu}p^2}{2-d}+\frac{d\eta^{\mu\nu}p^2}{2-d}+2p^\mu p^\nu-\eta^{\mu\nu}p^2\right)\epsilon_\nu(p)$$

$$= \frac{4ie^2}{(4\pi)^{d/2}}\epsilon^*_\mu(p)\frac{\Gamma(\frac{\epsilon}{2})\Gamma(\frac{d}{2})^2}{\Gamma(d)}(-p^2)^{-\frac{\epsilon}{2}}\left(-2\eta^{\mu\nu}p^2+2p^\mu p^\nu\right)\epsilon_\nu(p)$$

$$= \frac{8ie^2}{(4\pi)^{d/2}}\epsilon^*_\mu(p)\frac{\Gamma(\frac{\epsilon}{2})\Gamma(\frac{d}{2})^2}{\Gamma(d)}(-p^2)^{-\frac{\epsilon}{2}+1}\left(\eta^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right)\epsilon_\nu(p)$$

$$= \epsilon^*_\mu(p)\,\alpha\,\frac{8i}{(4\pi)^{d/2-1}}\frac{\Gamma(\frac{\epsilon}{2})\Gamma(\frac{d}{2})^2}{\Gamma(d)}(-p^2)^{1-\frac{\epsilon}{2}}\left(\eta^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right)\epsilon_\nu(p)$$

$$= \epsilon^*_\mu(p)\,\Pi^{\mu\nu}(p^2)\,\epsilon_\nu(p)$$

We need $\Pi^{\mu\nu}(p^2)\propto\left(\eta^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right)$ to fulfill the Ward identity. Physically, this is just current conservation which arises as a consequence of the gauge symmetry of QED. With $\Pi^{\mu\nu}(p^2)\propto\left(\eta^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right)$, we get

$$p_\mu\Pi^{\mu\nu}(p^2)\propto p_\mu\left(\eta^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right)=p^\nu-\frac{p^2p^\nu}{p^2}=0.$$