Quantum Field Theory II – Final Exam

Devised by Tilman Plehn, Advanced QFT

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Problem 1: One-loop calculations in QED

1) QED with massless fermion field $\Psi(x)$ and massless gauge field $A_\mu(x)$:

$$\mathcal{L} = -\frac{1}{4} F_{0,\mu\nu} F_0^{\mu\nu} + i\overline{\Psi}_0 \not{D}_0 \Psi_0, \quad D_0^\mu = \partial^\mu + ie_0 A_0^\mu.$$

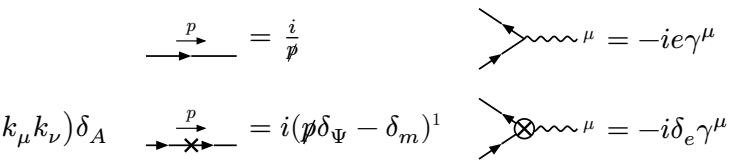
We use the Feynman rules to derive the one-loop corrections to the fermion propagator $i\mathcal{M}[f(p) \to f(p)]$ of a fermion f with mom. p. To apply renormalized perturbation theory, we split the bare Lagrangian into renormalized quantities, e.g. $\Psi = Z_\Psi^{-1/2}\Psi_0$ and $A^\mu = Z_A^{-1/2} A_0^\mu$, and counterterms as follows

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\overline{\Psi}\not{\partial}\Psi - eA_\mu \overline{\Psi}\gamma^\mu\Psi - \frac{1}{4}\delta_A F_{\mu\nu}F^{\mu\nu} + i\delta_\Psi \overline{\Psi}\not{\partial}\Psi - \delta_e A_\mu \overline{\Psi}\gamma^\mu\Psi,$$

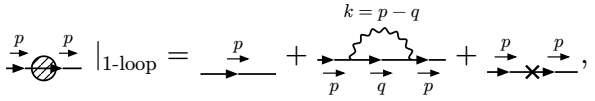
where $\delta_A = Z_A - 1$, $\delta_\Psi = Z_\Psi - 1$ and $\delta_e = e_0 Z_A^{1/2} Z_\Psi - e$. Integrating by parts, the photonic counterterm yields

$$\begin{aligned}
\delta_\gamma &= \int \mathrm{d}^4x \left(-\frac{1}{4}\delta_A F_{\mu\nu}F^{\mu\nu}\right) = -\frac{1}{4}\delta_A \int \mathrm{d}^4x \left(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu - \partial_\nu A_\mu \partial^\mu A^\nu + \partial_\nu A_\mu \partial^\nu A^\mu\right) \\
&= -\frac{1}{2}\delta_A \int \mathrm{d}^4x \left(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu\right) \\
&= -\frac{1}{2}\delta_A \int \mathrm{d}^4x \left(-A^\nu \partial_\mu \partial^\mu A_\nu + A_\nu \partial_\mu \partial^\nu A^\mu\right) \\
&= -\frac{1}{2}\delta_A \int \mathrm{d}^4x\, A_\nu (-\partial^2 g^{\mu\nu} + \partial^\mu\partial^\nu) A_\mu.
\end{aligned}$$

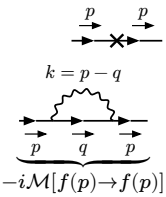
So our complete set of Feynman rules yields



At one loop, the fermion propagator receives the following contributions



of which only the latter two are corrections. We compute



$$= \overline{u}(p)(-ie\gamma^\nu) \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{i\not{q}}{q^2} \frac{-ig_{\mu\nu}}{k^2} (-ie\gamma^\mu) u(p)$$

[1] The δ_m term needs to be added even in massless theories.

$$
\begin{aligned}
&= -e^2\overline{u}(p)\int\frac{\mathrm{d}^dq}{(2\pi)^d}\,\frac{\gamma_\mu \not{q}\gamma^\mu}{q^2(p-q)^2}\,u(p), \quad \gamma_\mu \not{q}\gamma^\mu = \gamma_\mu(-\gamma^\mu \not{q} + 2q^\mu) = -2\not{q} \\
&= -e^2\overline{u}(p)\,\frac{\Gamma(1+1)}{\Gamma(1)\Gamma(1)}\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^dq}{(2\pi)^d}\,\frac{-2\not{q}}{\underbrace{(x(p-q)^2+(1-x)q^2)^2}_{xp^2-2xpq+xq^2+q^2-xq^2}}\,u(p) \\
&= 2e^2\overline{u}(p)\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^dq}{(2\pi)^d}\,\frac{\not{q}}{((\underbrace{q-xp}_{l})^2+x(1-x)p^2)^2}\,u(p) \\
&= 8\pi\alpha_{\mathrm{em}}\overline{u}(p)\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^dl}{(2\pi)^d}\,\frac{\not{l}^{0\text{ (odd)}}+x\not{p}}{\left(l^2+x(1-x)p^2\right)^2}\,u(p) \\
&= 8\pi\alpha_{\mathrm{em}}\overline{u}(p)\not{p}F(p^2)u(p), \quad (\not{p}u(p)=0)
\end{aligned}
$$

where we calculate $F(p^2)$ using the subst. $z = \frac{\Delta}{l_E^2-\Delta}$, $l_E = \Delta^{1/2}\big(\frac{1}{z}-1\big)^{1/2}$:

$$
\begin{aligned}
F(p^2) &= i\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^dl_E}{(2\pi)^d}\,\frac{x}{\big[l_E^2-\underbrace{x(1-x)p^2}_{\Delta}\big]^2} = i\int_0^1\mathrm{d}x\,x\int\frac{\mathrm{d}\Omega_d}{(2\pi)^d}\int_0^\infty\mathrm{d}l_E\,\frac{l_E^{d-1}}{\left(l_E^2-\Delta\right)^2} \\
&= i\int_0^1\mathrm{d}x\,x\,\frac{2\pi^{d/2}}{(2\pi)^d\Gamma\big(\frac{d}{2}\big)}\int_1^0\mathrm{d}z\,\frac{\left(l_E^2-\Delta\right)^2}{-2\Delta l_E}\,\frac{l_E^{d-1}}{\left(l_E^2-\Delta\right)^2} \\
&= \frac{i}{(4\pi)^{d/2}\Gamma\big(\frac{d}{2}\big)}\int_0^1\mathrm{d}x\,x\int_0^1\mathrm{d}z\,\Delta^{\frac{d}{2}-2}z^{1-\frac{d}{2}}(1-z)^{\frac{d}{2}-1} \\
&= \frac{i}{(4\pi)^{d/2}\Gamma\big(\frac{d}{2}\big)}\,\frac{\Gamma\big(2-\frac{d}{2}\big)\Gamma\big(\frac{d}{2}\big)}{\Gamma\big(2-\frac{d}{2}+\frac{d}{2}\big)}\int_0^1\mathrm{d}x\,x\Delta^{\frac{d}{2}-2} = \frac{i\Gamma\big(\frac{\varepsilon}{2}\big)}{(4\pi)^{d/2}}\int_0^1\mathrm{d}x\,x\big(x(1-x)p^2\big)^{\frac{d}{2}-2}.
\end{aligned}
$$

To bring out the divergence, we expand the integrand,

$$
\begin{aligned}
\int_0^1\mathrm{d}x\,x\big(x(1-x)p^2\big)^{-\varepsilon/2} &= \int_0^1\mathrm{d}x\,x\Big(1-\frac{\varepsilon}{2}\ln\big(x(1-x)p^2\big)+\mathcal{O}(\varepsilon^2)\Big) \\
&= \frac{1}{2}-\frac{\varepsilon}{2}\int_0^1\mathrm{d}x\,x\ln\big(x(1-x)p^2\big)+\mathcal{O}(\varepsilon^2).
\end{aligned}
$$

We can also expand the prefactors $\Gamma\big(\frac{\varepsilon}{2}\big) = \frac{2}{\varepsilon}-\gamma+\mathcal{O}(\varepsilon)$ and

$$
\frac{1}{(4\pi)^{d/2}} = \frac{1}{(4\pi)^{2-\varepsilon/2}} = \frac{(4\pi)^{\varepsilon/2}}{16\pi^2} = \frac{1}{16\pi^2}\Big(1+\frac{\varepsilon}{2}\ln(4\pi)+\mathcal{O}(\varepsilon^2)\Big).
$$

Assembling everything, we get

$$
\begin{aligned}
F(p^2) &= \frac{i}{16\pi^2}\Big(1+\frac{\varepsilon}{2}\ln(4\pi)+\mathcal{O}(\varepsilon^2)\Big)\Big(\frac{2}{\varepsilon}-\gamma+\mathcal{O}(\varepsilon)\Big)\left(\frac{1}{2}-\frac{\varepsilon}{2}\int_0^1\mathrm{d}x\,x\ln\big(x(1-x)p^2\big)+\mathcal{O}(\varepsilon^2)\right) \\
&= \frac{i}{32\pi^2}\left(\frac{2}{\varepsilon}-\gamma+\ln(4\pi)-2J(p^2)\right)+\mathcal{O}(\varepsilon), \quad J(p^2) \coloneqq \int_0^1\mathrm{d}x\,x\ln\big(x(1-x)p^2\big).
\end{aligned}
$$

Including the counterterm, the amputated 1PI two-point function at one loop is thus

$$
\Sigma(\not{p}) = i\not{p}\left[\delta_\Psi+\frac{e^2}{16\pi^2}\left(\frac{2}{\varepsilon}-\gamma+\ln(4\pi)-2J(p^2)\right)\right]-i\delta_m.
$$

If we choose our renormalization conditions as

$$\Sigma(\not{p})|_{\not{p}=\mu} = 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}\not{p}}\Sigma(\not{p})|_{\not{p}=\mu} = 0,$$

we obtain, using $\mathrm{d}J/\,\mathrm{d}\not{p} = \int_0^1 \mathrm{d}x\, x\,\frac{2\not{p}}{p^2} = \frac{1}{\not{p}}$ and the product rule, a counterterm δ_Ψ of

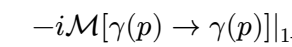
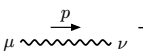
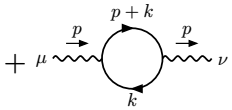
$$\frac{\mathrm{d}}{\mathrm{d}\not{p}}\Sigma(\not{p})|_{\not{p}=\mu} = i\left[\delta_\Psi + \frac{e^2}{16\pi^2}\left(\frac{2}{\varepsilon} - \gamma + \ln(4\pi) - 2J(\mu^2) - 2\right)\right] = 0$$

$$\implies \delta_\Psi = -\frac{e^2}{16\pi^2}\left(\frac{2}{\varepsilon} - \gamma + \ln(4\pi) - 2J(\mu^2) - 2\right)$$

and for δ_m

$$\Sigma(\not{p})|_{\not{p}=\mu} = \frac{2ie^2\mu}{16\pi^2} - i\delta_m = 0 \quad \implies \quad \delta_m = \frac{e^2\mu}{8\pi^2}.$$

2. We calculate the one-loop corrections to the photon propagator, again using the Feynman rules of renormalized perturbation theory.

$-i\mathcal{M}[\gamma(p)\to\gamma(p)]|_{\text{1-loop}} =$  $|_{\text{1-loop}} =$  $+$ 

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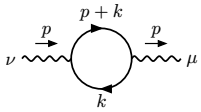
The first correction gives

 $= \varepsilon^*_\mu(p)\big(-ie(\gamma^\mu)^A_{\ B}\big)\displaystyle\int \frac{\mathrm{d}^dk}{(2\pi)^d}\,\frac{i(\not{p}+\not{k})^B_{\ C}}{(p+k)^2}\,\frac{i(\not{k})^C_{\ D}}{k^2}\big(-ie(\gamma^\nu)^D_{\ A}\big)\varepsilon_\nu(p)$

$$= -e^2\varepsilon^*_\mu(p)\int\frac{\mathrm{d}^dk}{(2\pi)^d}\,\frac{\operatorname{tr}(\gamma^\mu(\not{p}+\not{k})\gamma^\nu\not{k})}{k^2(p+k)^2}\,\varepsilon_\nu(p)$$

$$\begin{aligned}\operatorname{tr}(\gamma^\mu(\not{p}+\not{k})\gamma^\nu\not{k}) &= p_\rho k_\sigma \operatorname{tr}(\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma) + \operatorname{tr}\big(\gamma^\mu\not{k}k_\rho(2\eta^{\rho\nu} - \gamma^\rho\gamma^\nu)\big)\\ &= p_\rho k_\sigma 4(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\mu\sigma}\eta^{\rho\nu}) + 2\operatorname{tr}(\gamma^\mu\not{k}k^\nu) - k^2\operatorname{tr}(\gamma^\mu\gamma^\nu)\\ &= 4(p^\mu k^\nu - p^\sigma k_\sigma\eta^{\mu\nu} + p^\nu k^\mu) + 2k_\rho k^\nu\operatorname{tr}(\gamma^\mu\gamma^\rho) - k^2 4\eta^{\mu\nu}\\ &= 4(p^\mu k^\nu - p\cdot k\eta^{\mu\nu} + p^\nu k^\mu) + 8k^\mu k^\nu - 4k^2\eta^{\mu\nu}\end{aligned}$$

Using a Feynman parametrization to simplify the denominator,

$$i\mathcal{M}[\gamma(p)\to\gamma(p)] = -e^2\varepsilon^*_\mu(p)\int\frac{\mathrm{d}^dk}{(2\pi)^d}\int_0^1\mathrm{d}x\,\frac{\operatorname{tr}(\gamma^\mu(\not{p}+\not{k})\gamma^\nu\not{k})}{\left[x(p+k)^2 + (1-x)k^2\right]^2}\,\varepsilon_\nu(p),$$

$$\begin{aligned}x(p+k)^2 + (1-x)k^2 &= xp^2 + 2xpk + xk^2 + k^2 - xk^2\\ &= (k+xp)^2 + x(1-x)p^2 =: l^2 + x(1-x)p^2 = l^2 - \Delta,\end{aligned}$$

where $l = k + xp$, $\mathrm{d}l = \mathrm{d}k$.

$$
\begin{aligned}
i\mathcal{M} &= -e^2\varepsilon_\mu^*(p)\,\mathrm{tr}(\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma)\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d l}{(2\pi)^d}\,\frac{\overbrace{(l_\rho+(1-x)p_\rho)(l_\sigma - xp_\sigma)}^{l_\rho l_\sigma - x(1-x)p_\rho p_\sigma,\ \text{other terms odd}}}{\left(l^2-\Delta\right)^2}\,\varepsilon_\nu(p) \\
&= -e^2\varepsilon_\mu^*(p)\,\mathrm{tr}(\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma)\int_0^1 \mathrm{d}x\left[\frac{-i\Gamma\left(1-\frac{d}{2}\right)}{(4\pi)^{d/2}}\,\frac{\eta_{\rho\sigma}}{2}\,\Delta^{\frac{d}{2}-1} - x(1-x)p_\rho p_\sigma\,\frac{i\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{d/2}}\,\Delta^{\frac{d}{2}-2}\right]\varepsilon_\nu(p),
\end{aligned}
$$

where we used A.44 and A.46 from Peskin & Schroeder to perform the l-integration. We simplify

$$
\begin{aligned}
i\mathcal{M} &= \frac{ie^2\varepsilon_\mu^*}{(4\pi)^{d/2}}\,\Gamma\left(2-\frac{d}{2}\right)\mathrm{tr}(\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma)\int_0^1 \mathrm{d}x\left(\frac{\eta_{\rho\sigma}}{2-d}\left(-x(1-x)p^2\right)^{\frac{d}{2}-1} + x(1-x)p_\rho p_\sigma\left(-x(1-x)p^2\right)^{\frac{d}{2}-2}\right)\varepsilon_\nu(p) \\
&= \frac{ie^2}{(4\pi)^{d/2}}\,\varepsilon_\mu^*(p)\Gamma\left(\frac{\varepsilon}{2}\right)\mathrm{tr}(\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma)\left(\frac{-p^2\eta_{\rho\sigma}}{2-d} + p_\rho p_\sigma\right)\left(-p^2\right)^{\frac{d}{2}-2}\int_0^1 \mathrm{d}x\left[x(1-x)\right]^{\frac{d}{2}-1}\varepsilon_\nu(p) \\
&= \frac{ie^2}{(4\pi)^{d/2}}\,\varepsilon_\mu^*(p)\Gamma\left(\frac{\varepsilon}{2}\right)\mathrm{tr}(\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma)\left(-p^2\right)^{-\varepsilon/2}\left(\frac{-p^2}{2-d}\eta_{\rho\sigma} + p_\rho p_\sigma\right)\frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma(d)}\,\varepsilon_\nu(p) \\
&= \frac{ie^2}{(4\pi)^{d/2}}\,\varepsilon_\mu^*(p)\,\frac{\Gamma\left(\frac{\varepsilon}{2}\right)\Gamma\left(\frac{d}{2}\right)^2}{\Gamma(d)}\left(-p^2\right)^{-\varepsilon/2}4(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\mu\sigma}\eta^{\rho\nu})\left(\frac{-p^2}{2-d}\eta_{\rho\sigma} + p_\rho p_\sigma\right)\varepsilon_\nu(p) \\
&= \frac{4ie^2}{(4\pi)^{d/2}}\,\varepsilon_\mu^*(p)\,\frac{\Gamma\left(\frac{\varepsilon}{2}\right)\Gamma\left(\frac{d}{2}\right)^2}{\Gamma(d)}\left(-p^2\right)^{-\varepsilon/2}\left(\frac{-p^2}{2-d}\left(\eta^{\mu\nu} - \eta^{\mu\nu}\underbrace{\eta^\rho_\rho}_{d} + \eta^{\mu\nu}\right) + \left(p^\mu p^\nu - \eta^{\mu\nu}p^2 + p^\mu p^\nu\right)\right)\varepsilon_\nu(p) \\
&= \frac{4ie^2}{(4\pi)^{d/2}}\,\varepsilon_\mu^*(p)\,\frac{\Gamma\left(\frac{\varepsilon}{2}\right)\Gamma\left(\frac{d}{2}\right)^2}{\Gamma(d)}\left(-p^2\right)^{-\varepsilon/2}\left(\frac{-2\eta^{\mu\nu}p^2}{2-d} + \frac{d\eta^{\mu\nu}p^2}{2-d} + 2p^\mu p^\nu - \eta^{\mu\nu}p^2\right)\varepsilon_\nu(p) \\
&= \frac{4ie^2}{(4\pi)^{d/2}}\,\varepsilon_\mu^*(p)\,\frac{\Gamma\left(\frac{\varepsilon}{2}\right)\Gamma\left(\frac{d}{2}\right)^2}{\Gamma(d)}\left(-p^2\right)^{-\varepsilon/2}(-2\eta^{\mu\nu}p^2 + 2p^\mu p^\nu)\varepsilon_\nu(p) \\
&= \frac{8ie^2}{(4\pi)^{d/2}}\,\varepsilon_\mu^*(p)\,\frac{\Gamma\left(\frac{\varepsilon}{2}\right)\Gamma\left(\frac{d}{2}\right)^2}{\Gamma(d)}\left(-p^2\right)^{1-\varepsilon/2}\left(\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\varepsilon_\nu(p) \\
&= \varepsilon_\mu^*(p)\alpha\,\frac{8i}{(4\pi)^{d/2-1}}\,\frac{\Gamma\left(\frac{\varepsilon}{2}\right)\Gamma\left(\frac{d}{2}\right)^2}{\Gamma(d)}\left(-p^2\right)^{1-\varepsilon/2}\left(\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\varepsilon_\nu(p) \\
&= \varepsilon_\mu^*(p)\Pi^{\mu\nu}(p^2)\varepsilon_\nu(p).
\end{aligned}
$$

We need $\Pi^{\mu\nu}(p^2) \propto \left(\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)$ to fulfill the Ward identity. Physically, this is just current conservation which arises as a consequence of the gauge symmetry of QED. With $\Pi^{\mu\nu}(p^2) \propto \left(\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)$, we get

$$
p_\mu\Pi^{\mu\nu}(p^2) \propto p_\mu\left(\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) = p^\nu - \frac{p^2 p^\nu}{p^2} = 0.
$$