Exam on Quantum Field Theory I

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Problem 1 (12 points) – Quantization of a complex scalar field

A free complex-valued scalar field is described by the Lagrangian density

$$\mathcal{L} = \partial_\mu \varphi^\dagger \partial^\mu \varphi - m^2 \varphi^\dagger \varphi. \tag{1}$$

a) Show that the above Lagrangian is invariant under the global phase transformation $\varphi \to e^{i\alpha}\varphi$. What do we call these transformations in the language of field theory? What are the physical implications of such invariances?

$$\varphi \to e^{i\alpha}\varphi \quad \Longrightarrow \quad \varphi^\dagger \to \left(e^{i\alpha}\varphi\right)^\dagger = \varphi^\dagger e^{-i\alpha},$$

$$\mathcal{L} \to \mathcal{L}' = \partial_\mu \varphi^\dagger e^{-i\alpha} \partial^\mu e^{i\alpha}\varphi - m^2 \varphi^\dagger e^{-i\alpha} e^{i\alpha}\varphi = \partial_\mu \varphi^\dagger \partial^\mu \varphi - m^2 \varphi^\dagger \varphi = \mathcal{L}.$$

Such a global phase transformation is classified as $U(1)$. Underlying the Lagrangian's invariance is a continuous symmetry. According to Noether's theorem such symmetries invoke conserved charges, i.e. physical quantities.

b) Find the conjugate momenta associated to each component.

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}(x)} = \dot{\varphi}^\dagger(x), \quad \pi^\dagger(x) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^\dagger(x)} = \dot{\varphi}(x). \tag{2}$$

c) Show that the corresponding Hamiltonian H can be cast as

$$\begin{aligned} H &= \int \mathrm{d}^3x\, \mathcal{H}(x) = \int \mathrm{d}^3x \left(\pi(x)\dot{\varphi}(x) + \pi^\dagger(x)\dot{\varphi}^\dagger(x) - \mathcal{L}(x)\right) \\ &= \int \mathrm{d}^3x \left(\pi(x)\pi^\dagger(x) + \nabla\varphi^\dagger(x)\nabla\varphi(x) + m^2\varphi^\dagger(x)\varphi(x)\right). \end{aligned} \tag{3}$$

$$\begin{aligned} \pi(x)\dot{\varphi}(x) + \pi^\dagger(x)\dot{\varphi}^\dagger(x) - \mathcal{L}(x) &= \dot{\varphi}^\dagger\dot{\varphi} + \dot{\varphi}\dot{\varphi}^\dagger - \dot{\varphi}^\dagger\dot{\varphi} + \nabla\varphi^\dagger\nabla\varphi + m^2\varphi^\dagger\varphi \\ &= \pi^\dagger\pi + \nabla\varphi^\dagger\nabla\varphi + m^2\varphi^\dagger\varphi. \end{aligned}$$

d) The corresponding quantized fields we can write as

$$\varphi(x) = \int \tilde{\mathrm{d}}k \left(a(k)e^{-ikx} + b^\dagger(k)e^{ikx}\right), \quad \varphi^\dagger(x) = \int \tilde{\mathrm{d}}k \left(a^\dagger(k)e^{ikx} + b(k)e^{-ikx}\right).$$

With these expressions at hand, verify that the Hamiltonian (3) takes on the diagonal form

$$H = \int \mathrm{d}^3x\, \mathcal{H}(x) = \int \tilde{\mathrm{d}}k\, E_k \big(a^\dagger(k)a(k) + b^\dagger(k)b(k)\big).$$

The time and space derivatives of the fields are

$$\dot{\varphi}(x) = -i \int \tilde{\mathrm{d}}k\, E_k \big(a(k)e^{-ikx} - b^\dagger(k)e^{ikx}\big), \qquad \dot{\varphi}^\dagger(x) = i \int \tilde{\mathrm{d}}k\, E_k \big(a^\dagger(k)e^{ikx} - b(k)e^{-ikx}\big),$$

$$\nabla\varphi(x) = i\int \tilde{\mathrm{d}}k\,\vec{k}\big(a(k)e^{-ikx} - b^\dagger(k)e^{ikx}\big), \qquad \nabla\varphi^\dagger(x) = -i\int \tilde{\mathrm{d}}k\,\vec{k}\big(a^\dagger(k)e^{ikx} - b(k)e^{-ikx}\big).$$

Inserting them into the Hamiltonian, the x-integration sets $p = \pm k$ and leaves a factor $(2\pi)^3\delta^{(3)}(...) = 2E_k(2\pi)^3\delta^{(3)}(...)/(2E_k)$, i.e. a factor $1/(2E_k)$ inside $\int\tilde{\mathrm{d}}k$. With $E_k^2 = \vec{k}^2 + m^2$, the terms oscillating in time cancel and we get

$$\begin{aligned}
H &= \int \mathrm{d}^3x\,\mathcal{H}(x) = \int \mathrm{d}^3x\big(\dot\varphi\dot\varphi^\dagger + \nabla\varphi^\dagger\nabla\varphi + m^2\varphi^\dagger\varphi\big)\\
&= \int \mathrm{d}^3x \int \tilde{\mathrm{d}}k\,E_k\big(a(k)e^{-ikx} - b^\dagger(k)e^{ikx}\big)\int \tilde{\mathrm{d}}p\,E_p\big(a^\dagger(p)e^{ipx} - b(p)e^{-ipx}\big)\\
&\quad + \int \mathrm{d}^3x \int \tilde{\mathrm{d}}k\,\vec{k}\big(a(k)e^{-ikx} - b^\dagger(k)e^{ikx}\big)\int \tilde{\mathrm{d}}p\,\vec{p}\big(a^\dagger(p)e^{ipx} - b(p)e^{-ipx}\big)\\
&\quad + m^2\int \mathrm{d}^3x \int \tilde{\mathrm{d}}k\,\big(a(k)e^{-ikx} + b^\dagger(k)e^{ikx}\big)\int \tilde{\mathrm{d}}p\,\big(a^\dagger(p)e^{ipx} + b(p)e^{-ipx}\big)\\
&= \int \tilde{\mathrm{d}}k\,\frac{E_k}{2}\big(a(k)a^\dagger(k) - a(k)b(-k)e^{-2iE_kt} - b^\dagger(k)a^\dagger(-k)e^{2iE_kt} + b^\dagger(k)b(k)\big)\\
&\quad + \int \tilde{\mathrm{d}}k\,\frac{\vec{k}^2}{2E_k}\big(a(k)a^\dagger(k) + a(k)b(-k)e^{-2iE_kt} + b^\dagger(k)a^\dagger(-k)e^{2iE_kt} + b^\dagger(k)b(k)\big)\\
&\quad + \int \tilde{\mathrm{d}}k\,\frac{m^2}{2E_k}\big(a(k)a^\dagger(k) + a(k)b(-k)e^{-2iE_kt} + b^\dagger(k)a^\dagger(-k)e^{2iE_kt} + b^\dagger(k)b(k)\big)\\
&= \int \tilde{\mathrm{d}}k\,E_k\big(a(k)a^\dagger(k) + b^\dagger(k)b(k)\big) = \int \tilde{\mathrm{d}}k\,E_k\big(a^\dagger(k)a(k) + (2\pi)^3 2E_k\delta^{(3)}(0) + b^\dagger(k)b(k)\big)\\
&= \int \tilde{\mathrm{d}}k\,E_k\big(a^\dagger(k)a(k) + b^\dagger(k)b(k)\big) + 2(2\pi)^3\int \tilde{\mathrm{d}}k\,E_k^2\delta^{(3)}(0),
\end{aligned}$$

The last term is the (infinite) zero-point energy, which is dropped by normal ordering.