

Exam on Quantum Field Theory I

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Problem 1 (12 points) - Quantization of a complex scalar field

A free complex-valued scalar field is described by the Lagrangian density

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi. \quad (1)$$

a) Show that the above Lagrangian is invariant under the global phase transformation $\phi \rightarrow e^{i\alpha} \phi$. What do we call these transformations in the language of field theory? What are the physical implications of such invariances?

$$\phi \rightarrow e^{i\alpha} \phi \Rightarrow \phi^\dagger \rightarrow (e^{i\alpha} \phi)^\dagger = \phi^\dagger e^{-i\alpha}$$

$$\mathcal{L} \rightarrow \mathcal{L}' = \partial_\mu \phi^\dagger e^{-i\alpha} \partial^\mu e^{i\alpha} \phi - m^2 \phi^\dagger e^{-i\alpha} e^{i\alpha} \phi = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi = \mathcal{L}$$

Such a global phase transformation is classified as $U(1)$. Underlying the Lagrangian's invariance is a continuous symmetry. According to Noether's theorem such symmetries invoke conserved charges, i.e. physical quantities.

b) Find the conjugate momenta associated to each component.

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)} = \dot{\phi}^\dagger(x), \quad \pi^\dagger(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^\dagger(x)} = \dot{\phi}(x). \quad (2)$$

c) Show that the corresponding Hamiltonian H can be cast as

$$\begin{aligned} H &= \int d^3x \mathcal{H}(x) = \int d^3x \left(\pi(x) \dot{\phi}(x) + \pi^\dagger(x) \dot{\phi}^\dagger(x) - \mathcal{L}(x) \right) \\ &= \int d^3x \left(\pi(x) \pi^\dagger(x) + \vec{\nabla} \phi^\dagger(x) \vec{\nabla} \phi(x) + m^2 \phi^\dagger(x) \phi(x) \right) \end{aligned} \quad (3)$$

$$\begin{aligned} \pi(x) \dot{\phi}(x) + \pi^\dagger(x) \dot{\phi}^\dagger(x) - \mathcal{L}(x) &= \dot{\phi}^\dagger \dot{\phi} + \dot{\phi} \dot{\phi}^\dagger - \dot{\phi}^\dagger \dot{\phi} + \vec{\nabla} \phi^\dagger \vec{\nabla} \phi + m^2 \phi^\dagger \phi \\ &= \pi^\dagger \pi + \vec{\nabla} \phi^\dagger \vec{\nabla} \phi + m^2 \phi^\dagger \phi \end{aligned}$$

d) The corresponding quantized fields we can write as

$$\phi(x) = \int \tilde{d}k \left(a(k) e^{-ikx} + b^\dagger(k) e^{ikx} \right), \quad \phi^\dagger(x) = \int \tilde{d}k \left(a^\dagger(k) e^{ikx} + b(k) e^{-ikx} \right)$$

With these expressions at hand, verify that the Hamiltonian (3) takes on the diagonal form

$$H = \int d^3x \mathcal{H}(x) = \int d^3k E_k \left(a^\dagger(k) a(k) + b^\dagger(k) b(k) \right)$$

$$\dot{\phi}(x) = -i \int \tilde{d}k E_k \left(a(k) e^{-ikx} - b^\dagger(k) e^{ikx} \right), \quad \dot{\phi}^\dagger(x) = i \int \tilde{d}k E_k \left(a^\dagger(k) e^{ikx} - b(k) e^{-ikx} \right)$$

$$\vec{\nabla} \phi(x) = i \int \tilde{d}k \vec{k} \left(a(k) e^{-ikx} - b^\dagger(k) e^{ikx} \right), \quad \vec{\nabla} \phi^\dagger(x) = -i \int \tilde{d}k \vec{k} \left(a^\dagger(k) e^{ikx} - b(k) e^{-ikx} \right)$$

$$H = \int d^3x \mathcal{H}(x) = \int d^3x \left(\dot{\phi} \dot{\phi}^\dagger + \vec{\nabla} \phi \cdot \vec{\nabla} \phi^\dagger + m^2 \phi \phi^\dagger \right)$$

$$= \int d^3x \int \tilde{d}k E_k \left(a(k) e^{-ikx} - b^\dagger(k) e^{ikx} \right) \int \tilde{d}p E_p \left(a^\dagger(p) e^{ipx} - b(p) e^{-ipx} \right)$$

$$+ \int d^3x \int \tilde{d}k \vec{k} \left(a(k) e^{-ikx} - b^\dagger(k) e^{ikx} \right) \int \tilde{d}p \vec{p} \left(a^\dagger(p) e^{ipx} - b(p) e^{-ipx} \right)$$

$$+ m^2 \int d^3x \int \tilde{d}k \left(a(k) e^{-ikx} + b^\dagger(k) e^{ikx} \right) \int \tilde{d}p \left(a^\dagger(p) e^{ipx} + b(p) e^{-ipx} \right)$$

$$= \int \tilde{d}k E_k^2 \left(a(k) a^\dagger(k) - a(k) b^\dagger(k) e^{-2iE_k x^0} - b^\dagger(k) a^\dagger(k) e^{2iE_k x^0} + b^\dagger(k) b(k) \right)$$

$$+ \int \tilde{d}k k^2 \left(a(k) a^\dagger(k) + a(k) b^\dagger(k) e^{-2iE_k x^0} + b^\dagger(k) a^\dagger(k) e^{2iE_k x^0} + b^\dagger(k) b(k) \right)$$

$$+ \int \tilde{d}k m^2 \left(a(k) a^\dagger(k) + a(k) b^\dagger(k) e^{-2iE_k x^0} + b^\dagger(k) a^\dagger(k) e^{2iE_k x^0} + b^\dagger(k) b(k) \right)$$

$$= 2 \int \tilde{d}k E_k^2 \left(a(k) a^\dagger(k) + b^\dagger(k) b(k) \right) = 2 \int \tilde{d}k E_k^2 \left(a^\dagger(k) a(k) + (2\pi)^3 2E_k \delta^{(4)}(0) (a + b^\dagger(k)) (k) \right)$$

$$= 2 \int \tilde{d}k E_k^2 \left(a^\dagger(k) a(k) + b^\dagger(k) b(k) \right) + 4(2\pi)^3 \int \tilde{d}k E_k^3 \delta^{(4)}(0)$$