

Exam on Quantum Field Theory I

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Problem 1 (12 points) – Quantization of a complex scalar field

A free complex-valued scalar field is described by the Lagrangian density

$$\mathcal{L} = \partial_\mu \varphi^\dagger \partial^\mu \varphi - m^2 \varphi^\dagger \varphi. \quad (1)$$

a) Show that the above Lagrangian is invariant under the global phase transformation $\varphi \rightarrow e^{i\alpha} \varphi$. What do we call these transformations in the language of field theory? What are the physical implications of such invariances?

$$\varphi \rightarrow e^{i\alpha} \varphi \implies \varphi^\dagger \rightarrow (e^{i\alpha} \varphi)^\dagger = \varphi^\dagger e^{-i\alpha},$$

$$\mathcal{L} \rightarrow \mathcal{L}' = \partial_\mu \varphi^\dagger e^{-i\alpha} \partial^\mu e^{i\alpha} \varphi - m^2 \varphi^\dagger e^{-i\alpha} e^{i\alpha} \varphi = \partial_\mu \varphi^\dagger \partial^\mu \varphi - m^2 \varphi^\dagger \varphi = \mathcal{L}.$$

Such a global phase transformation is classified as $U(1)$. Underlying the Lagrangian's invariance is a continuous symmetry. According to Noether's theorem such symmetries invoke conserved charges, i.e. physical quantities.

b) Find the conjugate momenta associated to each component.

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}(x)} = \dot{\varphi}^\dagger(x), \quad \pi^\dagger(x) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^\dagger(x)} = \dot{\varphi}(x). \quad (2)$$

c) Show that the corresponding Hamiltonian H can be cast as

$$\begin{aligned} H &= \int d^3x \mathcal{H}(x) = \int d^3x (\pi(x) \dot{\varphi}(x) + \pi^\dagger(x) \dot{\varphi}^\dagger(x) - \mathcal{L}(x)) \\ &= \int d^3x (\pi(x) \pi^\dagger(x) + \nabla \varphi^\dagger(x) \nabla \varphi(x) + m^2 \varphi^\dagger(x) \varphi(x)). \end{aligned} \quad (3)$$

$$\begin{aligned} \pi(x) \dot{\varphi}(x) + \pi^\dagger(x) \dot{\varphi}^\dagger(x) - \mathcal{L}(x) &= \dot{\varphi}^\dagger \dot{\varphi} + \dot{\varphi} \dot{\varphi}^\dagger - \dot{\varphi}^\dagger \dot{\varphi} + \nabla \varphi^\dagger \nabla \varphi + m^2 \varphi^\dagger \varphi \\ &= \pi^\dagger \pi + \nabla \varphi^\dagger \nabla \varphi + m^2 \varphi^\dagger \varphi. \end{aligned}$$

d) The corresponding quantized fields we can write as

$$\varphi(x) = \int \tilde{d}k (a(k) e^{-ikx} + b^\dagger(k) e^{ikx}), \quad \varphi^\dagger(x) = \int \tilde{d}k (a^\dagger(k) e^{ikx} + b(k) e^{-ikx}).$$

With these expressions at hand, verify that the Hamiltonian (3) takes on the diagonal form

$$H = \int d^3x \mathcal{H}(x) = \int \tilde{d}k E_k (a^\dagger(k) a(k) + b^\dagger(k) b(k)).$$

The time and space derivatives of the fields are

$$\dot{\varphi}(x) = -i \int \tilde{d}k E_k (a(k) e^{-ikx} - b^\dagger(k) e^{ikx}), \quad \dot{\varphi}^\dagger(x) = i \int \tilde{d}k E_k (a^\dagger(k) e^{ikx} - b(k) e^{-ikx}),$$

$$\nabla\varphi(x) = i \int \tilde{d}k \vec{k} (a(k)e^{-ikx} - b^\dagger(k)e^{ikx}), \quad \nabla\varphi^\dagger(x) = -i \int \tilde{d}k \vec{k} (a^\dagger(k)e^{ikx} - b(k)e^{-ikx}).$$

Inserting them into the Hamiltonian, the x -integration sets $p = \pm k$ and leaves a factor $(2\pi)^3\delta^{(3)}(\dots) = 2E_k(2\pi)^3\delta^{(3)}(\dots)/(2E_k)$, i.e. a factor $1/(2E_k)$ inside $\int \tilde{d}k$. With $E_k^2 = \vec{k}^2 + m^2$, the terms oscillating in time cancel and we get

$$\begin{aligned} H &= \int d^3x \mathcal{H}(x) = \int d^3x (\dot{\varphi}\dot{\varphi}^\dagger + \nabla\varphi^\dagger\nabla\varphi + m^2\varphi^\dagger\varphi) \\ &= \int d^3x \int \tilde{d}k E_k (a(k)e^{-ikx} - b^\dagger(k)e^{ikx}) \int \tilde{d}p E_p (a^\dagger(p)e^{ipx} - b(p)e^{-ipx}) \\ &\quad + \int d^3x \int \tilde{d}k \vec{k} (a(k)e^{-ikx} - b^\dagger(k)e^{ikx}) \int \tilde{d}p \vec{p} (a^\dagger(p)e^{ipx} - b(p)e^{-ipx}) \\ &\quad + m^2 \int d^3x \int \tilde{d}k (a(k)e^{-ikx} + b^\dagger(k)e^{ikx}) \int \tilde{d}p (a^\dagger(p)e^{ipx} + b(p)e^{-ipx}) \\ &= \int \tilde{d}k \frac{E_k}{2} (a(k)a^\dagger(k) - a(k)b(-k)e^{-2iE_k t} - b^\dagger(k)a^\dagger(-k)e^{2iE_k t} + b^\dagger(k)b(k)) \\ &\quad + \int \tilde{d}k \frac{\vec{k}^2}{2E_k} (a(k)a^\dagger(k) + a(k)b(-k)e^{-2iE_k t} + b^\dagger(k)a^\dagger(-k)e^{2iE_k t} + b^\dagger(k)b(k)) \\ &\quad + \int \tilde{d}k \frac{m^2}{2E_k} (a(k)a^\dagger(k) + a(k)b(-k)e^{-2iE_k t} + b^\dagger(k)a^\dagger(-k)e^{2iE_k t} + b^\dagger(k)b(k)) \\ &= \int \tilde{d}k E_k (a(k)a^\dagger(k) + b^\dagger(k)b(k)) = \int \tilde{d}k E_k (a^\dagger(k)a(k) + (2\pi)^3 2E_k \delta^{(3)}(0) + b^\dagger(k)b(k)) \\ &= \int \tilde{d}k E_k (a^\dagger(k)a(k) + b^\dagger(k)b(k)) + 2(2\pi)^3 \int \tilde{d}k E_k^2 \delta^{(3)}(0), \end{aligned}$$

The last term is the (infinite) zero-point energy, which is dropped by normal ordering.