

Summary of Advanced Quantum Field Theory

Janosh Riebesell, April 2016

Contents

1	Path integral quantization	1
1.1	Transition amplitudes and correlation functions	1
1.2	Generating functionals for correlation functions	2
1.3	Schwinger-Dyson equation	2
1.4	1PI effective action	3
1.5	Fermionic path integral	3
1.6	Executive summary of QFT	3
2	Renormalization	4
2.1	Superficial divergence	4
2.2	Renormalization of QED	5
2.3	Callan-Symanzik equation	5
2.4	Wilsonian interpretation	6
3	Quantisation of Yang-Mills theory	7
3.1	Classical Yang-Mills theory	7
3.2	Quantizing Yang-Mills theory	8
3.3	Faddeev-Popov ghosts	9
3.4	BRST symmetry and physical Hilbert space	10

1 Path integral quantization

1.1 Transition amplitudes and correlation functions

- The path integral provides a formulation of quantum theory equivalent to canonical quantization.
- A quantum mechanical transition amplitude $\langle q_f, t_f | q_i, t_i \rangle = \langle q_f | e^{-i\hat{H}(t_f-t_i)} | q_i \rangle$ can, by partitioning of the transition time $\delta t = \frac{t_f-t_i}{N}$ and insertion of complete sets of states $\mathbb{1} = \int_{\mathbb{R}} dq_k |q_k\rangle \langle q_k|$ between each partition, be expressed as

$$\langle q_f, t_f | q_i, t_i \rangle = \int_{q(t_i)=q_i}^{q(t_f)=q_f} \mathcal{D}q(t) \mathcal{D}p(t) e^{i \int_{t_i}^{t_f} dt L(p, q)}, \quad (1)$$

with $\int_{t_i}^{t_f} dt L(p, q) \equiv S[p, q]$ and $L(p, q) = p\dot{q} - H(p, q)$.

- Analytic continuation by rotating t onto the lower half-plane via $t \rightarrow t(1 - i\epsilon)$ followed by performing the momentum path integral as a Gaussian yields the **Feynman-Kac formula**

$$\langle q_f, t_f | q_i, t_i \rangle = \int_{q(t_i)=q_i}^{q(t_f)=q_f} \mathcal{D}q(t) e^{i \int_{t_i}^{t_f} dt L(q, \dot{q})}, \quad (2)$$

where the factor $C^N = \left(\frac{-im}{2\pi\delta t}\right)^{N/2}$ from completion of the square is absorbed into $\mathcal{D}q(t)$.

- The path integral for scalar *fields* $\varphi(x)$ (as opposed to particles) is very similar to (2),

$$\langle \varphi_f(x), t_f | \varphi_i(x), t_i \rangle = \int_{\varphi(x, t_i)=\varphi_i(x)}^{\varphi(x, t_f)=\varphi_f(x)} \mathcal{D}\varphi e^{i \int_{t_i}^{t_f} d^4x \mathcal{L}(\varphi)}. \quad (3)$$

The **master formula** for an n -point quantum correlation function reads

$$G(x_1, \dots, x_n) \equiv \langle \Omega | T \prod_{j=1}^n \hat{\varphi}(x_j) | \Omega \rangle = \lim_{t \rightarrow \infty \cdot (1-i\epsilon)} \frac{\int \mathcal{D}\varphi \prod_{j=1}^n \varphi(x_j) e^{i \int_{-t}^t d^4x \mathcal{L}(\varphi)}}{\int \mathcal{D}\varphi e^{i \int_{-t}^t d^4x \mathcal{L}(\varphi)}} \quad (4)$$

- Time ordering T inside the path integral is taken care of automatically.

1.2 Generating functionals for correlation functions

- The **generating functional** $Z[J]$ of Green's functions $G(x_1, \dots, x_n)$ for some source $J(x)$ reads

$$Z[J] = \int \mathcal{D}\varphi e^{iS[\varphi] + iJ \cdot \varphi}, \quad (5)$$

where the functional inner product is defined as $J \cdot \varphi = \int_{\mathbb{R}^{1,3}} d^4x J(x) \varphi(x)$. $Z[J]$ maps the function $\varphi(x)$ to a number in $\mathbb{C}$. It is called *generating* functional because

$$\frac{Z[J]}{Z[0]} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \left(\prod_{j=1}^n \int_{\mathbb{R}^{1,3}} d^4x_j J(x_j) \right) G(x_1, \dots, x_n). \quad (6)$$

- This can be solved for $G(x_1, \dots, x_n)$ using the tools of functional calculus:

$$G(x_1, \dots, x_n) = \frac{1}{Z[0]} \prod_{j=1}^n \frac{\delta}{i\delta J(x_j)} Z[J] \Big|_{J=0}. \quad (7)$$

- $Z[0]$ contains no external points and represents the **partition function** $Z[0] = e^{\sum_i V_i}$, where the sum runs over all vacuum bubbles V_i of the theory. Consequently, $\frac{Z[J]}{Z[0]}$ contains no vacuum bubbles.
- Counting of loops: A fully connected Feynman diagram with E external and I internal lines, V vertices, and L loops (number of unfixed momentum integrals) satisfies **Euler's formula**

$$L = I - V + 1. \quad (8)$$

For $\mathcal{L}_{\text{int}}(\varphi) = \frac{\lambda}{n!} \varphi^n(x)$ we have $E + 2I = nV$.¹ Inserting eq. (8) yields

$$(n-2)V = 2L + (E-2). \quad (9)$$

Hence for fixed E , an expansion in L corresponds to an expansion in V .

1.3 Schwinger-Dyson equation

- An advantage of the path-integral method is that symmetries are more transparent. It becomes clear that classical symmetries carry over to the quantum theory – but only provided the path integral measure $\mathcal{D}\varphi = \mathcal{D}\varphi'$ is invariant. In that case, the **Schwinger-Dyson equation**

$$\int \mathcal{D}\varphi \left(\frac{\delta S[\varphi]}{\delta \varphi(x)} + J(x) \right) e^{iS[\varphi] + iJ \cdot \varphi} = 0 \quad (10)$$

states the classical equation of motion $\frac{\delta S}{\delta \varphi} + J = 0$ (in presence of a source J), holds as an operator equation in the quantum theory, i.e. inside the path integral (provided $\nexists$ contact terms $\propto \delta(x - x_j)$).

- For a continuous global classical symmetry $\varphi \rightarrow \varphi' = \varphi + \delta\varphi$ with conserved Noether current $j^\mu(x)$ given by $\frac{\delta S}{\delta \varphi(x)} \delta\varphi(x) = -\partial_\mu j^\mu(x) = 0$, acting on eq. (10) with $\prod_{j=1}^n \frac{\delta}{i\delta J(x_j)}$ and taking $J = 0$ afterwards gives the **Ward-Takahashi identity**, i.e. the statement of current conservation up to contact terms inside correlation functions,

¹Every vertex connects to n lines, while every external line connects to one and every internal line to two vertices.

$$\partial_\mu \left\langle \Omega \left| T j^\mu \prod_{j=1}^n \varphi(x_j) \right| \Omega \right\rangle = -i \sum_{j=1}^n \langle \Omega | T \varphi(x_1) \dots \varphi(x_{j-1}) [\delta \varphi(x) \delta(x - x_j)] \varphi(x_{j+1}) \dots \varphi(x_n) | \Omega \rangle. \quad (11)$$

Like eq. (10), the Ward-Takahashi identity only holds for classical symmetries of $S[\varphi]$ that leave the measure invariant. If $D\varphi$ is affected, the symmetry is anomalous and current conservation (up to contact terms) does not hold at the quantum level.

1.4 1PI effective action

- $G(x_1, \dots, x_n)$ receives contributions from partially connected Feynman diagrams. As established by the LSZ formalism, only fully connected Greens functions $G^c(x_1, \dots, x_n)$ containing Feynman diagrams which do not factor into subdiagrams, enter the computation of scattering amplitudes.
- The generating functional of $G^c(x_1, \dots, x_n)$ is denoted $iW[J]$ (not to be confused with the effective action Γ below). It is closely related to $Z[J]$ via $\frac{Z[J]}{Z[0]} = e^{iW[J]}$.
- An important subclass of fully connected Feynman diagrams are the 1-particle-irreducible (1PI) diagrams, which cannot be cut into two non-trivial diagrams by cutting a single (internal) line. These are generated by the **1PI effective action** $\Gamma[\phi]$ defined as the Legendre transform of $W[J]$,

$$\Gamma[\phi] = W[J] - \phi \cdot J, \quad (12)$$

where $\phi(x) \equiv \frac{\delta W[J]}{\delta J(x)} = \langle \Omega | \hat{\varphi}(x) | \Omega \rangle_J$, and we assumed there to be a bijection between J and ϕ .²

- $\Gamma[\phi]$ and $S[\phi]$ are the same functionals at tree-level, i.e. $\Gamma[\phi] = S[\phi] + K[\phi]$ for some $K[\phi]$ starting at one-loop. $\frac{\delta \Gamma[\phi]}{\delta \phi(x)} = -J(x)$ for $J(x) = 0$ yields the **quantum effective equation of motion**.

1.5 Fermionic path integral

- Fermionic *anticommutation* relations $\{\hat{\psi}, \hat{\psi}^\dagger\} = 1$, $\{\hat{\psi}, \hat{\psi}\} = \{\hat{\psi}^\dagger, \hat{\psi}^\dagger\} = 0$ can be implemented using anticommuting (nilpotent) Grassmann-valued fields $\psi(x)$ out of a **Grassmann algebra** $\mathbb{A}$.
- The **path integral for fermionic fields** $\psi(x)$, $\bar{\psi}(x) = \psi^\dagger(x)\gamma^0$ takes the form

$$\langle \psi_f(x), t_f | \psi_i(x), t_i \rangle = \int_{\psi(x, t_i) = \psi_i(x)}^{\psi(x, t_f) = \psi_f(x)} \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{i \int_{t_i}^{t_f} d^4x \mathcal{L}(\psi, \bar{\psi})}, \quad (13)$$

where $\mathcal{L}(\psi, \bar{\psi}) = \bar{\psi}(x) [i\gamma^\mu \partial_\mu - m_0] \psi(x) + \mathcal{L}_{\text{int}}$. The four $n \times n$ -gamma-matrices (one for every spacetime dimension) span the Clifford algebra $C\ell^n(\mathbb{C})$ defined by the anticommutator $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \mathbb{1}_n$ with $n = 2^{d/2} = 4$. To project initial and final states to the vacuum $|\Omega\rangle$, the trick $m_0 \rightarrow m_0 - i\epsilon$ can be used (just like in the bosonic case).

- The **generating functional for fermionic correlators** is defined as

$$Z[\eta, \bar{\eta}] = \langle \Omega | T e^{i \int_{\mathbb{R}^{1,3}} d^4x [\mathcal{L}(\psi, \bar{\psi}) + \bar{\psi}(x)\eta(x) + \bar{\eta}(x)\psi(x)]} | \Omega \rangle = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{iS[\psi, \bar{\psi}] + i\bar{\psi} \cdot \eta + i\bar{\eta} \cdot \psi}, \quad (14)$$

with the external sources $\bar{\eta}(x)$, $\eta(x)$ as Grassmann-valued classical fields.

1.6 Executive summary of QFT

- Start with a classical action $S[\varphi]$ in which the field $\varphi(x)$ arises as the continuum limit $N \rightarrow \infty$ of a system of N harmonic oscillators.
- In the classical limit $\hbar \rightarrow 0$, $\varphi(x)$ is a definite function given by the classical equation of motion $\delta S[\varphi]/\delta \varphi(x) = 0$. For $\hbar$ finite, quantum fluctuations arise. These are encoded in $Z[J]$, where the path integral takes into account all possible functions $\varphi(x)$ could assume.

²In the Euclidean theory $W_E[J]$ is convex and such a one-to-one correspondence between sources and φ -v.e.v.s exists.

- What we can compute in the quantum theory are correlation functions. In particular the quantum expectation value of the field $\varphi(x)$ in the presence of a source J is

$$\phi_J(x) \equiv \langle \varphi(x) \rangle_J \equiv \langle \Omega | \hat{\varphi}(x) | \Omega \rangle_J = \frac{1}{Z[J]} \int \mathcal{D}\varphi \varphi(x) e^{-S[\varphi] + \varphi \cdot J}. \quad (15)$$

- With $Z[J] = e^{-W[J]}$ (in Euclidean signature), we can compute this as

$$\phi_J(x) = -\frac{\delta W[J]}{\delta J(x)}. \quad (16)$$

- In terms of the Legendre transform $\Gamma[\phi]$ of $W[J]$, we have

$$\frac{\delta \Gamma[\phi]}{\delta \phi(x)} = J(x). \quad (17)$$

By integrating out quantum fluctuations $f(x) = \varphi(x) - \phi(x)$, $\Gamma[\phi]$ gives a quantum effective action

$$e^{-\Gamma[\phi]} = \frac{1}{Z[0]} \int \mathcal{D}f e^{-S[\phi+f] + \frac{\delta \Gamma}{\delta \phi} \cdot f}. \quad (18)$$

Replacing $S[\varphi]$ by $\Gamma[\phi]$ introduces 1PI amputated vertices and fully resummed propagators. Thus, computing at tree-level with $\Gamma[\phi]$ already gives the full quantum theory!

2 Renormalization

2.1 Superficial divergence

- For a scalar theory in d dimensions with (bare) Lagrangian $\mathcal{L}_0 = \frac{1}{2}(\partial\varphi)^2 - \frac{m_0^2}{2}\varphi^2 - \frac{\lambda_0}{n!}\varphi^n$, the naive UV structure of a diagram $\mathcal{D}$ with L loops $\propto \int_{\mathbb{R}^d} d^d k$ and I propagators $\propto (k^2 - m^2)^{-1}$ is

$$\mathcal{D} \xrightarrow{k \rightarrow \infty} \frac{\int_{\mathbb{R}^d} d^d k_1 \dots \int_{\mathbb{R}^d} d^d k_L}{k_1^2 \dots k_I^2}. \quad (19)$$

The **superficial degree of divergence** D of $\mathcal{D}$ is defined as the difference in powers of momentum between numerator and denominator, i.e. $D = dL - 2I$.

- Regularizing the divergence with a momentum cutoff Λ , i.e. $\int_{-\infty}^{\infty} dk \rightarrow \lim_{\Lambda \rightarrow \infty} \int_{-\Lambda}^{\Lambda} dk$, diagrams fall into three categories of UV behavior: 1. $D > 0 \Rightarrow \mathcal{D} \propto \Lambda^D$ (superficially divergent). 2. $D < 0 \Rightarrow \mathcal{D} \propto \Lambda^{-|D|}$ (superf. finite). 3. $D = 0 \Rightarrow \mathcal{D} \propto \ln(\Lambda)$ (superf. log-divergent).
- The actual UV behavior may differ from the superficial one for three reasons: 1. For $D \geq 0$, a diagram may still be finite if symmetry constrains the amplitude or leads to cancellations among infinite terms. 2. For $D < 0$, a diagram may still be divergent if it contains a divergent subdiagram. 3. Tree-level diagrams have $D = 0$, but are finite.
- For $\mathcal{L}_0$ as above, D depends on the mass dimension of the coupling $[\lambda_0] = d - \frac{d-2}{2}n$ as

$$D = d - [\lambda_0]V - \frac{d-2}{2}E. \quad (20)$$

The UV properties of a theory are decisively determined by (the sign of) the prefactor of V .

1. If $D \propto +V$, there exists an infinite number of superficially divergent amplitudes since for every E , diagrams with high enough V diverge. The theory is thus **non-renormalizable** $\Leftrightarrow [\lambda_0] < 0$.
2. If $D \propto -V$ (and $d \geq 2$), only a finite number of diagrams is divergent but divergences appear *at every loop-order*. Such theories are called **renormalizable** and arise for $[\lambda_0] = 0$.

3. If $D \propto -V$, for high-enough loop order, all diagrams become superf. finite, making the theory **super-renormalizable** $\Leftrightarrow [\lambda_0] > 0$.
 - E.g. for $n = d = 4$, we have $[\lambda_0] = 0$ and $D = 4 - E$ independent of L or V . Hence, φ^4 -theory in $d = 4$ is renormalizable with only three superficially divergent amplitudes, $E = 0, 2, 4$ (at every loop-order).
- By the **BPHZ theorem**, (power counting) renormalizability is sufficient for a theory to maintain predictivity. (The non-trivial aspect of this theorem concerns the complete cancellation of divergent subdiagrams by counterterms of the previous loop-order.)

2.2 Renormalization of QED

- **Regularization** is the practice of isolating divergences. The three common methods in QFT are
 1. **Cutoff reg.** regularizes divergent momentum integrals via $\int_{-\infty}^{\infty} dk \rightarrow \lim_{\Lambda \rightarrow \infty} \int_{-\Lambda}^{\Lambda} dk$. However, this is inconsistent with the Ward identities and gauge invariance because transformations of the sort $A^\mu \rightarrow A^\mu + \partial^\mu \alpha(x)$ cannot be carried out at the cutoff, making it a useless method in QED.
 2. **Dimensional reg.** (used most often) evaluates divergent integrals in $d = 4 - \epsilon$ dimensions. The result is expanded in powers of ϵ which isolates the divergence as a pole as $\epsilon \rightarrow 0$.
 3. **Pauli-Villars reg.** takes a divergent diagram and subtracts from it the same diagram but with a fictitious massive particle in the loop, e.g. a photon of mass Λ . This removes the divergence because for $k \rightarrow \infty$, the mass in the loop becomes irrelevant and both diagrams asymptote to the same value. But for $\Lambda \rightarrow \infty$, the auxiliary diagram vanishes and we recover the actual process.
- The QED Lagrangian with symmetry group $U(1)$ describes the coupling of spin-1/2 bispinor fields $\psi(x)$ (electron, positron) to a covariant spin-1 gauge field $A_\mu(x)$ (photon) generated by the transformation behavior of the spinors themselves. $\mathcal{L}_{\text{QED}}$ can be expressed i.t.o. the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and the covariant derivative $D_\mu = \partial_\mu + ieA_\mu$ as

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi = -\frac{1}{4} F^2 + \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - A_\mu j^\mu, \quad (21)$$

where m is the fermion mass, e is the coupling constant equal to the (electric) charge of the bispinor field and $j^\mu = e\bar{\psi}\gamma^\mu\psi$ is the conserved fermion current associated with the $U(1)$ symmetry.

- A QED diagram with E_e (E_γ) external fermions (photons) has superficial degree of divergence

$$D = 4 - \frac{3}{2} E_e - E_\gamma. \quad (22)$$

Since $[e] = 0$, QED is renormalizable with seven superficially (four actually³) divergent diagrams.

2.3 Callan-Symanzik equation

- Renormalization automatically introduces a mass scale μ – the **renormalization scale** – into the quantum theory via the renormalization conditions (even when the classical theory was scale-free).
- To quantify the dependence of coupling constants on the renormalization scale μ , we can study the **Callan-Symanzik** (or **renormalization group**) equation (here for massive φ^4 -theory)

$$(\mu\partial_\mu + \beta_\lambda\partial_\lambda + \beta_{m^2}\partial_{m^2} + n \cdot \gamma_\varphi)G_n(x_1, \dots, x_n) = 0, \quad (23)$$

³The symmetries at work preventing some of QED's superficial divergences are discrete charge conjugation $j^\mu \rightarrow -j^\mu$, $A^\mu \rightarrow -A^\mu$, chiral symmetry (arises for $m = 0$), and the Ward identity $k^\mu \mathcal{D}_\mu = 0$ for a diagram $\mathcal{D} = \xi^\mu \mathcal{D}_\mu$ involving an external photon of momentum k^μ ($k^2 = 0$) and polarization ξ^μ .

where $\beta_\lambda = \mu \frac{d\lambda}{d\mu} \Big|_{\lambda_0, m_0}$, $\beta_{m^2} = \mu \frac{dm^2}{d\mu} \Big|_{\lambda_0, m_0}$, and $\gamma_\varphi = \frac{\mu}{2} \frac{d \ln(Z)}{d\mu} \Big|_{\lambda_0, m_0} \cdot \beta_\lambda$ for example describes how the physical coupling λ changes as we change the energy scale μ at which we perform an experiment.

- The CS equation allows us to (perturbatively) compute β_λ , β_{m^2} , γ_φ explicitly by first computing $G_n(x_1, \dots, x_n)$ and then plugging it into (23). E.g. $\beta_\lambda = \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3)$ for massless φ^4 -theory.
- The change in $\lambda(\mu)$ as we change μ is called **renormalisation group flow** or **running coupling**. $\lambda(\mu)$ gives the strength of the interaction at energy scale μ .
- Depending on the sign of β there are three qualitatively different **RG behaviors**.
 1. If $\beta(\lambda) > 0$, $\lambda(\mu)$ increases as μ increases. If we start with a perturbative value λ_0 at μ_0 and follow the RG flow for increasing μ , then at some scale, $\lambda(\mu)$ may cease to be perturbative. If by a non-perturbative analysis beyond that point one finds $\beta(\lambda) > 0 \forall \lambda$, then $\lambda(\mu)$ increases indefinitely. This can result in a divergent coupling $\lambda \rightarrow \infty$, either asymptotically as $\mu \rightarrow \infty$, or even for finite values of $\mu \rightarrow \mu_L$. The latter instance is referred to as a **Landau pole**.
 - One appears e.g. in QED at $\mu_L = \mu_0 \exp\left(\frac{3\pi}{2\alpha_0}\right)$ since

$$\alpha(\mu) = \frac{\alpha_0}{1 - \frac{2\alpha_0}{3\pi} \ln\left(\frac{\mu}{\mu_0}\right)} \xrightarrow{\mu \rightarrow \mu_L} \infty. \quad (24)$$

On the other hand, $\beta(\lambda) > 0$ means the theory is perturbatively well-defined in the infrared, where λ becomes small. If $\lambda \rightarrow 0$ as $\mu \rightarrow 0$, the theory even becomes free in the infrared. Such a non-interacting fixed point is called **Gaussian fixed point**.

2. If $\beta(\lambda) < 0$, $\lambda(\mu)$ decreases as μ increases. The theory is perturbative in the UV, but may cease to be perturbative in the IR. If $\lambda \rightarrow 0$ as $\mu \rightarrow \infty$, the theory becomes free in the UV. This is called **asymptotic freedom**.⁴
3. If $\beta = 0 \forall \mu$, λ is independent of μ . Such a theory is **conformal**, i.e. scale-independent. Since the counterterms do not induce any scale dependence, there cannot be any UV divergences altogether and the theory is UV finite.

2.4 Wilsonian interpretation

- The original understanding of renormalization was:
 - The cutoff Λ is merely a way to regulate divergent integrals without physical meaning.
 - Renormalization is a trick to remove the cutoff-dependence in physical amplitudes. This procedure allows us to take $\Lambda \rightarrow \infty$ without encountering divergences.
 - This comes at the cost of losing predictability for some physical masses and couplings.
 - In a renormalizable theory, only a finite number of such physical couplings must be taken as input parameters from experiment to end up with a well-defined (otherwise predictive) theory.
- The **Wilsonian approach** gives a different interpretation: We should think of QFT as an *effective description* accurate only for energies below an intrinsic cutoff Λ_0 . At energies beyond Λ_0 the field theory picture does not correctly model the microscopic degrees of freedom.⁵
 - The only known theory that is UV finite and asymptotes to a weakly coupled QFT in the infrared is **string theory**, which abandons the concept of pointlike particles, replacing them with excitations of a one-dimensional string of length ℓ_s . The string length is the intrinsic cutoff of the low-energy effective QFT. At distances near ℓ_s , the theory deviates from a regular field theory in that it becomes non-local, thus avoiding UV divergences and arbitrary input parameters.

⁴In $d = 4$, the only known example for asymptotic freedom is Yang-Mills theory.

⁵For example, QFT neglects gravity but all matter gravitates and gravity becomes non-negligible (compared to the other forces) near the Planck scale $M_{\text{Pl}} \approx 1/\sqrt{G_N} \approx 10^{19}$ GeV.

- Integrating out the degrees of freedom between the regulator Λ and an even smaller cutoff $\Lambda_0 < \Lambda$ yields the **Wilsonian effective action** S_W^{eff} . When computing correlators at scales below Λ_0 via S_W^{eff} , only momenta $|k| \leq \Lambda_0$ appear in the loops since all effects of the modes with $\Lambda_0 < |k| < \Lambda$ are already encoded in S_W^{eff} . This is not to be confused with the quantum effective action $\Gamma[\phi]$ which gives the full quantum theory already at tree-level. S_W^{eff} includes only those quantum effects due to the integrated-out modes k between $\Lambda_0 < |k| < \Lambda$ and loops must still be performed.
- Successive applications of this integration to a lower cutoff gives rise to the **renormalisation semi-group** (semi because we can only lower the cutoff; there does not exist an inverse operation).
- The running couplings in the Wilsonian picture are interpreted as the dependency of the couplings in S_W^{eff} on the cutoff. This identifies Λ_0 as the renormalization scale μ .

3 Quantisation of Yang-Mills theory

3.1 Classical Yang-Mills theory

- The gauge field $A_\mu(x)$ of a Yang-Mills theory with **non-Abelian Lie group** H^6 (whose elements are the gauge transformations that leave the theory invariant) takes values in the associated **Lie algebra** $\mathfrak{h}$. $\mathfrak{h}$ is a vector space equipped with a non-associative antisymmetric bilinear map $[\cdot, \cdot] : \mathfrak{h} \times \mathfrak{h} \rightarrow \mathfrak{h}$, the **Lie bracket**.
- Like any element of $\mathfrak{h}$, $A_\mu(x)$ can be expressed i.t.o. of a basis $\{T^a\}$, $a \in \{1, \dots, \dim(\mathfrak{h})\}$ of $\mathfrak{h}$ (that forms a complete set of generators of the underlying Lie group H):

$$A^\mu(x) = A^\mu_a(x) T^a \quad (\text{summation over } a \text{ implied}) \quad (25)$$

- The basis elements satisfy the Lie algebra's defining relation

$$[T^a, T^b] = i f^{ab}_c T^c, \quad (26)$$

i.t.o. the **structure constants** f^{ab}_c .⁷ Eq. (26) in turn fulfills the **Jacobi identity**

$$[[T^a, T^b], T^c] + [[T^b, T^c], T^a] + [[T^c, T^a], T^b] = 0. \quad (27)$$

- As an example, the Lie algebra $su(2)$ of dimension 3 has as one possible basis $T^i = \sigma_i/2$, with the three Pauli matrices σ_i , which generate the corresponding Lie group $SU(2)$. In this basis, the structure constants are given by the components of the Levi-Civita symbol ϵ^{ijk} .
- Every Lie algebra also possesses a symmetric bilinear form, the **Killing form**

$$\kappa^{ab} = T^a \circ T^b, \quad (28)$$

which is invariant under the **adjoint action** of the Lie group H ,

$$h T^a h^{-1} \circ h T^b h^{-1} = T^a \circ T^b \quad \forall h \in H. \quad (29)$$

- When working with H in matrix representation, e.g. $H = SU(N)$, the generators T^a are Hermitian traceless $N \times N$ -matrices and the Killing form $\circ$ acts simply as the trace on $\mathfrak{h}$,

$$T^a \circ T^b = \text{tr}_{\mathfrak{h}}(T^a T^b) = \frac{1}{2} \delta^{ab}. \quad (30)$$

⁶Generally, any compact, semi-simple Lie group will do, but most forms of Yang-Mills theory are based on $SU(N)$ with associated Lie algebra $\mathfrak{su}(N)$.

⁷The f^{ab}_c restrict the result of taking the Lie bracket of two generators T^a, T^b to a linear combination of all generators $\{T^c\}$, thereby determining the Lie brackets of all elements of $\mathfrak{h}$. This almost completely establishes the group structure of H , explaining the name structure constants.

The last equality only holds if H is compact as a manifold, in which case the Killing form is positive definite and can be suitably normalized.

- The Killing form κ^{ab} and its inverse⁸ $c\kappa_{ab}$ can be used to raise and lower Lie-algebra indices, e.g.

$$f^{abc} = f^{ab}{}_d \kappa^{dc}. \quad (31)$$

With all indices appearing on the same footing, the structure constants are totally antisymmetric and therefore invariant under cyclic permutations.⁹

- Under a **gauge transformation** $U \in H$ parametrized as $U(x) = e^{-ig\alpha(x)}$ with $g \in \mathbb{R}$ and $\alpha(x) \in \mathfrak{h}$ the gauge potential $A^\mu(x)$ transforms to linear order as

$$A^\mu(x) \rightarrow A^\mu(x) + D^\mu\alpha(x), \quad (32)$$

where the **adjoint covariant derivative** acts on $\mathfrak{h}$ -valued fields $\alpha(x)$,

$$D^\mu\alpha(x) \equiv \partial^\mu\alpha(x) + ig[A^\mu(x), \alpha(x)]. \quad (33)$$

- The associated **field strength tensor** of A^μ is given by

$$F_{\mu\nu}(x) = \frac{1}{ig} [D_\mu, D_\nu] = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + ig[A_\mu(x), A_\nu(x)]. \quad (34)$$

- $F_{\mu\nu}$ transforms under the adjoint action of H , $F_{\mu\nu} \rightarrow UF_{\mu\nu}U^{-1}$, and satisfies the Bianchi identity $D_\rho F_{\mu\nu} + D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} = 0$.
- I.t.o. the field strength $F_{\mu\nu}$, the gauge-invariant **Yang-Mills Lagrangian** can be written as

$$\mathcal{L}_{\text{YM}}(A) = -\frac{1}{2} \text{tr}_{\mathfrak{h}, \mathbb{R}^{1,3}}(F^2) = -\frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu}. \quad (35)$$

The commutator in $F_{\mu\nu}$ introduces cubic and quartic gauge field interactions into $\mathcal{L}_{\text{YM}}$. The gauge field's equation of motion is

$$D_\mu F^{\mu\nu} = 0 \quad \Leftrightarrow \quad \partial_\mu F^{\mu\nu} = -ig[A_\mu(x), F^{\mu\nu}]. \quad (36)$$

3.2 Quantizing Yang-Mills theory

- Gauge invariance as well as the fact that $A_0(x)$ appears without a time-derivative in $\mathcal{L}_{\text{YM}}$, i.e. is a non-dynamical field without conjugate momentum Π_0 , complicates the quantization of Yang-Mills theory. Variation of the action $S_{\text{YM}}[A]$ w.r.t. to $A_0(x)$ yields $D_i F^{0i} = 0$ ($i \in \{1, 2, 3\}$) which is itself a **non-dynamical constraint**, with $A_0(x)$ merely an unphysical Lagrange multiplier enforcing it.
- Canonical quantization of constrained systems requires special technology (e.g. the Gupta-Bleuler procedure for $U(1)$ gauge theories). Hence, path integral quantization is preferred for Yang-Mills.
- The naive path integral quantization of a gauge field A^μ proceeds by formulating an action $S[A]$, inverting the kinetic term $(K \cdot A)^\mu = -\partial^2 A^\mu + \partial^\mu \partial_\nu A^\nu$ to find the propagator $iD_F = K^{-1}$, and then perturbatively tackling the interacting theory. This runs into trouble because K is in fact *not* invertible due to its non-trivial kernel $\ker(K) \neq \{0\}$. Rather, $(K \cdot \partial\alpha) = 0 \ \forall \alpha(x) \in \mathfrak{h}$.¹⁰
- The cure is to remove the non-invertibility of K by excluding all but one element out of each set of gauge-equivalent field configurations related to first order by $A^\mu \rightarrow A^\mu + \partial^\mu\alpha$. Untruncated, a full gauge transformation is given by

⁸ c is merely a normalization factor determined by the structure of the Lie group H as a manifold.

⁹This is not so much naturally the case as up to our choice of basis matrices $\{T^a\}$. It can be proven that a basis always exists in which the f^{abc} have these properties.

¹⁰This problem is entirely due to gauge invariance and has nothing to do with the gauge group being Abelian or not.

$$A^\mu \rightarrow A_h^\mu = h A^\mu h^{-1} + \frac{i}{g} (\partial^\mu h) h^{-1}, \quad \text{for any } h \in H. \quad (37)$$

A^μ and A_h^μ are physically equivalent and lead the path integral to overcount because if one satisfies the e.o.m. so does the other. Given any A^μ , all equivalent field configurations lie in the same **orbit**

$$O_A = \{A_h^\mu \mid h \in H\}. \quad (38)$$

Hence, let $\mathcal{A}$ denote the space of *all* field configurations $A^\mu(x)$, then the physically inequivalent ones are captured precisely by the quotient space $\mathcal{A}/H$ which picks out exactly one field per orbit.

- Some path integral manipulations yield the **Yang-Mills partition function**

$$Z_{\text{YM}} = \int_{\mathcal{A}} \mathcal{D}A \delta[F(A)] \det(\Delta_{\text{FP}}) e^{iS_{\text{YM}}[A]}, \quad (39)$$

where the argument of the functional Dirac delta is the gauge fixing condition $F(A)$ which, given any field configuration $A^\mu(x) \in \mathcal{A}$ achieves $F(A^h) = 0$ (ideally) for exactly one unique $h \in H^{11}$, thereby effectively reducing the integration domain from $\mathcal{A}$ to $\mathcal{A}/H$. $\Delta_{\text{FP}} = -\frac{\partial F(A)}{\partial A^\mu} D^\mu$ is the **Faddeev-Popov matrix**. Eq. (39) can be used to calculate vacuum expectation values of any (gauge-invariant!) operator $\mathcal{O}(A) = \mathcal{O}(A^h)$ by the usual $t \rightarrow \infty(1 - i\epsilon)$ prescription (see eq. (4)).

3.3 Faddeev-Popov ghosts

- Introducing the $\mathfrak{h}$ -valued **Nakanishi-Lautrup auxiliary field** $B(x)$, we can rewrite $\delta[F(A)]$ as

$$\delta[F(A)] = \int \mathcal{D}B e^{i \int_{\mathbb{R}^{1,3}} d^4x B_a(x) F^a(x)}. \quad (40)$$

With the Grassmann- $\mathfrak{h}$ -valued **Faddeev-Popov ghost** $c(x)$ and **antighost** $\bar{c}(x)$, $\det(\Delta_{\text{FP}})$ becomes

$$\det(\Delta_{\text{FP}}) = \int \mathcal{D}\bar{c} \mathcal{D}c e^{i \int_{\mathbb{R}^{1,3}} d^4x \bar{c}_a(x) [\Delta_{\text{FP}} c(x)]^a}. \quad (41)$$

Inserting eqs. (40) and (41) into eq. (39), gives Z_{YM} as

$$Z_{\text{YM}} = \int \mathcal{D}A \mathcal{D}B \mathcal{D}\bar{c} \mathcal{D}c e^{iS[A, B, \bar{c}, c]}. \quad (42)$$

where the ABc action is

$$S[A, B, \bar{c}, c] = \int_{\mathbb{R}^{1,3}} d^4x \left(-\frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} + B_a(x) F^a(x) + \bar{c}_a(x) [\Delta_{\text{FP}} c(x)]^a \right). \quad (43)$$

The ghosts transform as scalar fields under $SO(1,3)$, but have fermionic statistics due to their Grassmannian nature. Thus they violate the spin-statistics theorem as well as unitarity since their Fock space does not have a positive definite norm.

¹¹The ideal case usually doesn't come to pass due to an irritating **residual gauge symmetry** that results in several gauge equivalent field configurations, so-called **Gribov copies**, which all fulfill $F(A) = 0$. Thus even the gauge-fixed path integral would still overcount if we did not restrict it to a fundamental domain where the gauge is unique.

3.4 BRST symmetry and physical Hilbert space

We are finished with the path integral and return to the formalism of canonical quantization from here on.

- The problem of determining the physical Hilbert space H_{phys} has two parts. We need to 1. guarantee a positive-definite norm on H_{phys} , and 2. show that time-evolution does not lead out of H_{phys} , i.e. the S-matrix needs to be a unitary operator on H_{phys} . If the criterion for a state to lie in H_{phys} is related to a symmetry of the full interacting theory, then invariance of the physical Hilbert space under time-evolution follows automatically because the S-matrix respects all symmetries.
- Note that the ABc action (43) possesses a **global residual fermionic symmetry** – the **BRST symmetry**. It is implemented via the Grassmann-odd nilpotent operator R acting as

$$RA^\mu = -D^\mu c = -(\partial^\mu c + ig[A^\mu, c]), \quad Rc = \frac{ig}{2}\{c, c\}, \quad R\bar{c} = -B, \quad RB = 0. \quad (44)$$

Defining $\psi(x) = \bar{c}^a \partial_\mu A_a^\mu + \frac{\xi}{2} \bar{c}^a B_a$, with the **gauge-fixing parameter** ξ , the ABc Lagrangian reads

$$\mathcal{L}(A, B, \bar{c}, c) = -\frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} - R\psi, \quad (45)$$

which fulfills $R\mathcal{L} = 0$, because for the first term, R is just a gauge transformation on A^μ that leaves F^2 invariant, and $R^2 = 0$ in the second.

- Like any quantum symmetry, the **BRST symmetry transformation**

$$\delta_\epsilon \Phi = \epsilon R\Phi, \quad \Phi \in \{A, B, \bar{c}, c\} \quad (46)$$

with ϵ a global Grassmann-valued parameter, is generated (in canonical quantization) by its associated **Noether charge** (operator) $\hat{Q}_{\text{BRST}}$ in the sense that

$$[\epsilon \hat{Q}_{\text{BRST}}, \hat{\Phi}] = i\delta_\epsilon \hat{\Phi}. \quad (47)$$

The **BRST charge** $\hat{Q}_{\text{BRST}}$ is conserved, nilpotent, and hermitian:

$$\hat{Q}_{\text{BRST}} = 0 \Leftrightarrow [\hat{H}, \hat{Q}_{\text{BRST}}] = 0, \quad \hat{Q}_{\text{BRST}}^2 = 0, \quad \hat{Q}_{\text{BRST}}^\dagger = \hat{Q}_{\text{BRST}}. \quad (48)$$

- Some useful mathematics for finding $\mathcal{H}_{\text{phys}}$ of quantum Yang-Mills theory: Any state $|\psi\rangle$ in a vector space $\mathcal{H}$ with a nilpotent linear operator $\hat{Q} : \mathcal{H} \rightarrow \mathcal{H}$ acting on it can be classified as either
 - $\hat{Q}$ -closed if $\hat{Q}|\psi\rangle = 0$, i.e. if $|\psi\rangle \in \ker(\hat{Q})$, or
 - $\hat{Q}$ -exact if $\exists |\chi\rangle \in \mathcal{H}$ such that $|\psi\rangle = \hat{Q}|\chi\rangle$, i.e. if $|\psi\rangle \in \text{Im}(\hat{Q})$.

Since for all $\hat{Q}$ -exact $|\psi\rangle$ it holds that $\hat{Q}|\psi\rangle = \hat{Q}^2|\chi\rangle = 0$, we have $\text{Im}(\hat{Q}) \subset \ker(\hat{Q})$. Further, the **$\hat{Q}$ -cohomology** is defined as the quotient space $\mathcal{C}(\hat{Q}) \equiv \ker(\hat{Q}) / \text{Im}(\hat{Q})$.

- Back to physics: The space of states on which the time-evolution operator $\hat{U} = \text{T}e^{-i\int_{\mathbb{R}} \hat{H} dt}$ is independent of the specific choice of gauge-fixing condition is given by $\ker(\hat{Q}_{\text{BRST}})$. Time evolution should not depend on gauge, so we require

$$|\psi\rangle \in \mathcal{H}_{\text{phys}} \quad \text{only if} \quad \hat{Q}_{\text{BRST}}|\psi\rangle = 0. \quad (49)$$

Within $\ker(\hat{Q}_{\text{BRST}})$, the BRST-exact states have zero overlap with all other states since $\forall |\psi\rangle \in \text{Im}(\hat{Q}_{\text{BRST}})$ with $|\psi\rangle = \hat{Q}_{\text{BRST}}|\chi\rangle$ and $\forall |\xi\rangle \in \ker(\hat{Q}_{\text{BRST}})$

$$\langle\psi|\xi\rangle = \langle\chi|\hat{Q}_{\text{BRST}}^\dagger|\xi\rangle = \langle\chi|\underbrace{\hat{Q}_{\text{BRST}}}_0|\xi\rangle = 0. \quad (50)$$

States with zero overlap with all other states can never be measured and are therefore unphysical. We thus define the **physical Hilbert space** of quantum Yang-Mills theory as the cohomology

$$\mathcal{H}_{\text{phys}} = \mathcal{C}(\hat{Q}_{\text{BRST}}) = \frac{\ker(\hat{Q}_{\text{BRST}})}{\text{Im}(\hat{Q}_{\text{BRST}})}. \quad (51)$$

- Evaluating the physical state condition eq. (49) reveals B -, $\bar{c}$ -, and c -excitations to be unphysical. The only physical states are of the form

$$|\psi\rangle \in \mathcal{H}_{\text{phys}} \quad \Leftrightarrow \quad |\psi\rangle = \xi_\mu |A^\mu(\mathbf{k})\rangle \quad \text{with } k^2 = 0 \text{ and } \xi_\mu k^\mu = 0. \quad (52)$$