Summary of
Advanced Quantum Field Theory

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April 2016

Contents

1 Path integral quantization

1.1 Transition amplitudes and correlation functions

- The path integral provides a formulation of quantum theory equivalent to canonical quantization.
- A quantum mechanical transition amplitude $\langle q_f, t_f | q_i, t_i\rangle = \langle q_f | e^{i\hat{H}(t_f - t_i)} | q_i, t_i\rangle$ can, by partitioning of the transition time $\delta t = \frac{t_f - t_i}{N}$ and insertion of complete sets of states $\mathbb{1} = \int_{\mathbb{R}} \mathrm{d}q_k \, |q_k\rangle\langle q_k|$ between each partition, be expressed as

$$\langle q_f, t_f | q_i, t_i\rangle = \int_{q(t_i)=q_i}^{q(t_f)=q_f} \mathcal{D}q(t)\,\mathcal{D}p(t)\, e^{i\int_{t_i}^{t_f} \mathrm{d}t\, L(p,q)}, \tag{1}$$

  with $\int_{t_i}^{t_f} \mathrm{d}t\, L(p,q) \equiv S[p,q]$ and $L(p,q) = p\dot{q} - H(p,q)$.

- Analytic continuation by rotating t onto the lower half-plane via $t \to t(1 - i\epsilon)$ followed by performing the momentum path integral as a Gaussian yields the **Feynman-Kac formula**

$$\langle q_f, t_f | q_i, t_i\rangle = \int_{q(t_i)=q_i}^{q(t_f)=q_f} \mathcal{D}q(t)\, e^{i\int_{t_i}^{t_f} \mathrm{d}t\, L(q,\dot{q})}, \tag{2}$$

where the factor $C^N = \big(\frac{-im}{2\pi\,\delta t}\big)^{N/2}$ from completion of the square is absorbed into $\mathcal{D}q(t)$.

- The path integral for scalar *fields* $\phi(x)$ (as opposed to particles) is very similar to (2),

$$\langle \phi_f(x), t_f | \phi_i(x), t_i \rangle = \int_{\phi(\boldsymbol{x},t_i)=\phi_i(\boldsymbol{x})}^{\phi(\boldsymbol{x},t_f)=\phi_f(\boldsymbol{x})} \mathcal{D}\phi \, e^{i \int_{t_i}^{t_f} \mathrm{d}^4x \, \mathcal{L}(\phi)}. \tag{3}$$

  The **master formula** for an n-point quantum correlation function reads

$$G(x_1, \ldots, x_n) \equiv \langle \Omega | \mathrm{T} \prod_{j=1}^{n} \hat{\phi}(x_j) | \Omega \rangle = \lim_{\substack{t\to\infty \\ \cdot(1-i\epsilon)}} \frac{\int \mathcal{D}\phi \, \prod_{j=1}^{n} \phi(x_j) \, e^{i\int_{-t}^{t} \mathrm{d}^4x \, \mathcal{L}(\phi)}}{\int \mathcal{D}\phi \, e^{i\int_{-t}^{t} \mathrm{d}^4x \, \mathcal{L}(\phi)}} \tag{4}$$

  – Time ordering T inside the path integral is taken care off automatically.

1.2 Generating functionals for correlation functions

- The **generating functional** $Z[J]$ of Green's functions $G(x_1, \ldots, x_n)$ for some source $J(x)$ reads

$$Z[J] = \int \mathcal{D}\phi \, e^{iS[\phi] + iJ\cdot\phi}, \tag{5}$$

  where the functional inner product is defined as $J \cdot \phi = \int_{\mathbb{R}^{1,3}} \mathrm{d}^4x \, J(x) \, \phi(x)$. $Z[J]$ maps the function $\phi(x)$ to a number in $\mathbb{C}$. It is called *generating* functional because

$$\frac{Z[J]}{Z[0]} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \bigg(\prod_{j=1}^{n} \int_{\mathbb{R}^{1,3}} \mathrm{d}^4x_j J(x_j) \bigg) \, G(x_1, \ldots, x_n). \tag{6}$$

- This can be solved for $G(x_1, \ldots, x_n)$ using the tools of functional calculus:

$$G(x_1, \ldots, x_n) = \frac{1}{Z[0]} \prod_{j=1}^{n} \frac{\delta}{i\delta J(x_j)} \, Z[J] \Big|_{J=0}. \tag{7}$$

- $Z[0]$ contains no external points and represents the **partition function** $Z[0] = e^{\sum_i V_i}$, where the sum runs over all vacuum bubbles V_i of the theory. Consequently, $\frac{Z[J]}{Z[0]}$ contains no vacuum bubbles.

- Counting of loops: A fully connected Feynman diagram with E external and I internal lines, V vertices, and L loops (number of unfixed momentum integrals) satisfies **Euler's formula**

$$L = I - V + 1. \tag{8}$$

  For $\mathcal{L}_{\text{int}}(\phi) = \frac{\lambda}{n!}\phi^n(x)$ we have $E + 2I = nV$.[1] Inserting eq. (8) yields

$$(n-2)V = 2L + (E-2). \tag{9}$$

  Hence for fixed E, an expansion in L corresponds to an expansion in V.

1.3 Schwinger-Dyson equation

- An advantage of the path-integral method is that symmetries are more transparent. It becomes clear that classical symmetries carry over to the quantum theory - but only provided the path integral measure $\mathcal{D}\phi = \mathcal{D}\phi'$ is invariant. In that case, the **Schwinger-Dyson equation**

$$\int \mathcal{D}\phi \Big(\frac{\delta S[\phi]}{\delta\phi(x)} + J(x) \Big) e^{iS[\phi] + iJ\cdot\phi} = 0 \tag{10}$$

  states the classical equation of motion $\frac{\delta S}{\delta \phi} + J = 0$ (in presence of a source J), holds as an operator equation in the quantum theory, i.e. inside the path integral (provided $\nexists$ contact terms $\propto \delta(x - x_j)$).

[1] Every vertex connects to n lines, while every external line connects to one and every internal line to two vertices.

- For a continuous global classical symmetry $\phi \to \phi' = \phi + \delta\phi$ with conserved Noether current $j^\mu(x)$ given by $\frac{\delta S}{\delta\phi(x)}\delta\phi(x) = -\partial_\mu j^\mu(x) = 0$, acting on eq. (10) with $\prod_{j=1}^n \frac{\delta}{i\delta J(x_j)}$ and taking $J=0$ afterwards gives the **Ward-Takahashi identity**, i.e. the statement of current conservation up to contact terms inside correlation functions,

$$\partial_\mu \Big\langle \Omega \Big| \mathrm{T} j^\mu \prod_{j=1}^n \phi(x_j) \Big| \Omega \Big\rangle = -i \sum_{j=1}^n \langle \Omega | \mathrm{T}\phi(x_1)\dots\phi(x_{j-1})\,[\delta\phi(x)\delta(x-x_j)]\,\phi(x_{j+1})\dots\phi(x_n)|\Omega\rangle. \quad (11)$$

  Like eq. (10), the Ward-Takahashi identity only holds for classical symmetries of $S[\phi]$ that leave the measure invariant. If $\mathcal{D}\phi$ is affected, the symmetry is anomalous and current conservation (up to contact terms) does not hold at the quantum level.

1.4 1PI effective action

- $G(x_1,\dots,x_n)$ receives contributions from partially connected Feynman diagrams. As established by the LSZ formalism, only fully connected Greens functions $G^c(x_1,\dots,x_n)$ containing Feynman diagrams which do not factor into subdiagrams, enter the computation of scattering amplitudes.

- The generating functional of $G^c(x_1,\dots,x_n)$ is called **effective action** and denoted $iW[J]$. It is closely related to $Z[J]$ via $\frac{Z[J]}{Z[0]} = e^{iW[J]}$.

- An important subclass of fully connected Feynman diagrams are the 1-particle-irreducible (1PI) diagrams, which cannot be cut into two non-trivial diagrams by cutting a single (internal) line. These are generated by the **1PI effective action** $\Gamma[\varphi]$ defined as the Legendre transform of $W[J]$,

$$\Gamma[\varphi] = W[J] - \varphi \cdot J, \quad (12)$$

  where $\varphi(x) \equiv \frac{\delta W[J]}{\delta J(x)} = \langle \Omega|\hat{\phi}(x)|\Omega\rangle_J$, and we assumed there to be a bijection between J and φ.[2]

- $\Gamma[\varphi]$ and $S[\varphi]$ are the same functionals at tree-level, i.e. $\Gamma[\varphi] = S[\varphi] + K[\varphi]$ for some $K[\varphi]$ starting at one-loop. $\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} = -J(x)$ for $J(x) = 0$ yields the **quantum effective equation of motion**.

1.5 Fermionic path integral

- Fermionic *anti*commutation relations $\{\hat{\boldsymbol{\psi}}, \hat{\boldsymbol{\psi}}^\dagger\} = 1$, $\{\hat{\boldsymbol{\psi}}, \hat{\boldsymbol{\psi}}\} = \{\hat{\boldsymbol{\psi}}^\dagger, \hat{\boldsymbol{\psi}}^\dagger\} = 0$ can be implemented using anticommuting (nilpotent) Grassmann-valued fields $\boldsymbol{\psi}(x)$ out of a **Grassmann algebra** $\mathbb{A}$.

- The **path integral for fermionic fields** $\boldsymbol{\psi}(x)$, $\bar{\boldsymbol{\psi}}(x) = \boldsymbol{\psi}^\dagger(x)\,\boldsymbol{\gamma}^0$ takes the form

$$\langle \boldsymbol{\psi}_f(\boldsymbol{x_f}), t_f | \boldsymbol{\psi}_f(\boldsymbol{x_i}), t_i \rangle = \int_{\boldsymbol{\psi}(\boldsymbol{x},t_i)=\boldsymbol{\psi}_i(\boldsymbol{x})}^{\boldsymbol{\psi}(\boldsymbol{x},t_f)=\boldsymbol{\psi}_f(\boldsymbol{x})} \mathcal{D}\bar{\boldsymbol{\psi}}\mathcal{D}\boldsymbol{\psi}\, e^{i\int_{t_i}^{t_f} \mathrm{d}^4x\, \mathcal{L}(\boldsymbol{\psi},\bar{\boldsymbol{\psi}})}, \quad (13)$$

  where $\mathcal{L}(\boldsymbol{\psi},\bar{\boldsymbol{\psi}}) = \bar{\boldsymbol{\psi}}(x)[i\boldsymbol{\gamma}^\mu\partial_\mu - m_0]\boldsymbol{\psi}(x) + \mathcal{L}_{\text{int}}$. The four $n \times n$-gamma-matrices (one for every spacetime dimension) span the Clifford algebra $C\ell^n(\mathbb{C})$ defined by the anticommutator $\{\boldsymbol{\gamma}^\mu, \boldsymbol{\gamma}^\nu\} = 2\,\eta^{\mu\nu}\mathbb{1}_n$ with $n = 2^{d/2} = 4$. To project initial and final states to the vacuum $|\Omega\rangle$, the trick $m_0 \to m_0 - i\epsilon$ can be used (just like in the bosonic case).

- The **generating functional for fermionic correlators** is defined as

$$Z[\boldsymbol{\eta}, \bar{\boldsymbol{\eta}}] = \langle \Omega | \mathrm{T} e^{i\int_{\mathbb{R}^{1,3}} \mathrm{d}^4x\,[\mathcal{L}(\boldsymbol{\psi},\bar{\boldsymbol{\psi}}) + \bar{\boldsymbol{\psi}}(x)\boldsymbol{\eta}(x) + \bar{\boldsymbol{\eta}}(x)\boldsymbol{\psi}(x)]} |\Omega\rangle = \int \mathcal{D}\bar{\boldsymbol{\psi}}\mathcal{D}\boldsymbol{\psi}\, e^{iS[\boldsymbol{\psi},\bar{\boldsymbol{\psi}}] + i\bar{\boldsymbol{\psi}}\cdot\boldsymbol{\eta} + i\bar{\boldsymbol{\eta}}\cdot\boldsymbol{\psi}}, \quad (14)$$

  with the external sources $\bar{\boldsymbol{\eta}}(x)$, $\boldsymbol{\eta}(x)$ as Grassmann-valued classical fields.

[2] In the Euclidean theory $W_E[J]$ is convex and such a one-to-one correspondence between sources and ϕ-v.e.v.s exists.

1.6 Executive summary of QFT

- Start with a classical action $S[\phi]$ in which the field $\phi(x)$ arises as the continuum limit $N \to \infty$ of a system of N harmonic oscillators.

- In the classical limit $\hbar \to 0$, $\phi(x)$ is a definite function given by the classical equation of motion $\delta S[\phi]/\delta\phi(x) = 0$. For $\hbar$ finite, quantum fluctuations arise. These are encoded in $Z[J]$, where the path integral takes into account all possible functions $\phi(x)$ could assume.

- What we can compute in the quantum theory are correlation functions. In particular the quantum expectation value of the field $\phi(x)$ in the presence of a source J is

$$\varphi_J(x) \equiv \langle \phi(x) \rangle_J \equiv \langle \Omega | \hat{\phi}(x) | \Omega \rangle_J = \frac{1}{Z[0]} \int \mathcal{D}\phi \, \phi(x) \, e^{-S[\phi] + \phi \cdot J}. \tag{15}$$

- With our definition of $W[J]$, we can compute this as

$$\varphi_J(x) = -\frac{\delta W[J]}{\delta J(x)}. \tag{16}$$

- In terms of the Legendre transform $\Gamma[\varphi]$ of $W[J]$, we have

$$\frac{\delta \Gamma[\varphi]}{\delta \varphi(x)} = J(x). \tag{17}$$

  By integrating out quantum fluctuations $f(x) = \phi(x) - \varphi(x)$, $\Gamma[\varphi]$ gives a quantum effective action

$$e^{-\Gamma[\varphi]} = \frac{1}{Z[0]} \int \mathcal{D}f \, e^{-S[\varphi + f] + \frac{\delta\Gamma}{\delta\varphi} \cdot f}. \tag{18}$$

  Replacing $S[\phi]$ by $\Gamma[\varphi]$ introduces 1PI amputated vertices and fully resummed propagators. Thus, computing at tree-level with $\Gamma[\varphi]$ already gives the full quantum theory!

2 Renormalization

2.1 Superficial divergence

- For a scalar theory in d dimensions with (bare) Lagrangian $\mathcal{L}_0 = \frac{1}{2}(\partial\phi)^2 - \frac{m_0^2}{2}\phi^2 - \frac{\lambda_0}{n!}\phi^n$, the naive UV structure of a diagram $\mathcal{D}$ with L loops $\propto \int_{\mathbb{R}^d} \mathrm{d}^d k$ and I propagators $\propto (k^2 - m^2)^{-1}$ is

$$\mathcal{D} \xrightarrow{k \to \infty} \frac{\int_{\mathbb{R}^d} \mathrm{d}^d k_1 \dots \int_{\mathbb{R}^d} \mathrm{d}^d k_L}{k_1^2 \dots k_I^2}. \tag{19}$$

  The **superficial degree of divergence** D of $\mathcal{D}$ is defined as the difference in powers of momentum between numerator and denominator, i.e. $D = dL - 2I$.

  - Regularizing the divergence with a momentum cutoff Λ, i.e. $\int_{-\infty}^{\infty} \mathrm{d}k \to \lim_{\Lambda\to\infty} \int_{-\Lambda}^{\Lambda} \mathrm{d}k$, diagrams fall into three categories of UV behavior: 1. $D > 0 \Rightarrow \mathcal{D} \propto \Lambda^D$ (superficially divergent). 2. $D < 0 \Rightarrow \mathcal{D} \propto \Lambda^{-|D|}$ (superf. finite). 3. $D = 0 \Rightarrow \mathcal{D} \propto \ln(\Lambda)$ (superf. log-divergent).
  - The actual UV behavior may differ from the superficial one for three reasons: 1. For $D \geq 0$, a diagram may still be finite if symmetry constrains the amplitude or leads to cancellations among infinite terms. 2. For $D < 0$, a diagram may still be divergent if it contains a divergent subdiagram. 3. Tree-level diagrams have $D = 0$, but are finite.

- For $\mathcal{L}_0$ as above, D depends on the mass dimension of the coupling $[\lambda_0] = d - \frac{d-2}{2}\, n$ as

$$D = d - [\lambda_0]\, V - \frac{d-2}{2}\, E. \tag{20}$$

  The UV properties of a theory are decisively determined by (the sign of) the prefactor of V.

1. If $D \propto +V$, there exists an infinite number of superficially divergent amplitudes since for every E, diagrams with high enough V diverge. The theory is thus **non-renormalizable** $\Leftrightarrow [\lambda_0] < 0$.
2. If $D \not\propto V$ (and $d \geq 2$), only a finite number of diagrams is divergent but divergences appear *at every loop-order*. Such theories are called **renormalizable** and arise for $[\lambda_0] = 0$.
3. If $D \propto -V$, for high-enough loop order, all diagrams become superf. finite, making the theory **super-renormalizable** $\Leftrightarrow [\lambda_0] > 0$.

– E.g. for $n = d = 4$, we have $[\lambda_0] = 0$ and $D = 4 - E$ independent of L or V. Hence, ϕ^4-theory in $d = 4$ is renormalizable with only three superficially divergent diagrams (at every loop-order).

- By the **BPHZ theorem**, (power counting) renormalizability is sufficient for a theory to maintain predictivity. (The non-trivial aspect of this theorem concerns the complete cancellation of divergent subdiagrams by counterterms of the previous loop-order.)

2.2 Renormalization of QED

- **Regularization** is the practice of isolating divergences. The three common methods in QFT are
  1. **Cutoff reg.** regularizes divergent momentum integrals via $\int_{-\infty}^{\infty} \mathrm{d}k \to \lim_{\Lambda\to\infty} \int_{-\Lambda}^{\Lambda} \mathrm{d}k$. However, this is inconsistent with the Ward identities and gauge invariance because transformations of the sort $A^\mu \to A^\mu + \partial^\mu \alpha(x)$ cannot be carried out at the cutoff, making it a useless method in QED.
  2. **Dimensional reg.** (used most often) evaluates divergent integrals in $d = 4 - \epsilon$ dimensions. The result is expanded in powers of ϵ which isolates the divergence as a pole as $\epsilon \to 0$.
  3. **Pauli-Villars reg.** takes a divergent diagram and subtracts from it the same diagram but with a fictitious massive particle in the loop, e.g. a photon of mass Λ. This removes the divergence because for $k \to \infty$, the mass in the loop becomes irrelevant and both diagrams asymptote to the same value. But for $\Lambda \to \infty$, the auxiliary diagram vanishes and we recover the actual process.

- The QED Lagrangian with symmetry group $U(1)$ describes the coupling of spin-1/2 bispinor fields $\boldsymbol{\psi}(x)$ (electron, positron) to a covariant spin-1 gauge field $A_\mu(x)$ (photon) generated by the transformation behavior of the spinors themselves. $\mathcal{L}_{\text{QED}}$ can be expressed i.t.o. the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and the covariant derivative $D_\mu = \partial_\mu + ieA_\mu$ as

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\boldsymbol{\psi}}(i\boldsymbol{\gamma}^\mu D_\mu - m)\boldsymbol{\psi} = -\frac{1}{4} F^2 + \bar{\boldsymbol{\psi}}(i\boldsymbol{\gamma}^\mu \partial_\mu - m)\boldsymbol{\psi} - A_\mu j^\mu, \tag{21}$$

  where m is the fermion mass, e is the coupling constant equal to the (electric) charge of the bispinor field and $j^\mu = e\bar{\boldsymbol{\psi}}\boldsymbol{\gamma}^\mu\boldsymbol{\psi}$ is the conserved fermion current associated with the $U(1)$ symmetry.

- A QED diagram with E_e (E_γ) external fermions (photons) has superficial degree of divergence

$$D = 4 - \frac{3}{2} E_e - E_\gamma. \tag{22}$$

  Since $[e] = 0$, QED is renormalizable with seven superficially (four actually[3]) divergent diagrams.

2.3 Callan-Symanzyk equation

- Renormalization automatically introduces a mass scale μ - the **renormalization scale** - into the quantum theory via the renormalization conditions (even when the classical theory was scale-free).

- To quantify the dependence of coupling constants on the renormalization scale μ, we can study the **Callan-Symanzik** (or **renormalization group**) equation (here for massive ϕ^4-theory)

$$\Big(\mu\partial_\mu + \beta_\lambda \partial_\lambda + \beta_{m^2} \partial_{m^2} + n \cdot \gamma_\phi\Big) G_n(x_1, \ldots, x_n) = 0, \tag{23}$$

[3] The symmetries at work preventing some of QED's superficial divergences are discrete charge conjugation $j^\mu \to -j^\mu$, $A^\mu \to -A^\mu$, chiral symmetry (arises for $m = 0$), and the Ward identity $k^\mu \mathcal{D}_\mu = 0$ for a diagram $\mathcal{D} = \xi^\mu \mathcal{D}_\mu$ involving an external photon of momentum k^μ ($k^2 = 0$) and polarization ξ^μ.

where $\beta_\lambda = \mu \frac{\mathrm{d}\lambda}{\mathrm{d}\mu}|_{\lambda_0,m_0}$, $\beta_{m^2} = \mu \frac{\mathrm{d}m^2}{\mathrm{d}\mu}|_{\lambda_0,m_0}$, and $\gamma_\phi = \frac{\mu}{2}\frac{\mathrm{d}\ln(Z)}{\mathrm{d}\mu}|_{\lambda_0,m_0}$. β_λ for example describes how the physical coupling λ changes as we change the energy scale μ at which we perform an experiment.

- The CS equation allows us to (perturbatively) compute β_λ, β_{m^2}, γ_ϕ explicitly by first computing $G_n(x_1, \dots, x_n)$ and then plugging it into (23). E.g. $\beta_\lambda = \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3)$ for massless ϕ^4-theory.
- The change in $\lambda(\mu)$ as we change μ is called **renormalisation group flow** or **running coupling**. $\lambda(\mu)$ gives the strength of the interaction at energy scale μ.

- Depending on the sign of β there are three qualitatively different **RG behaviors**.

1. If $\beta(\lambda) > 0$, $\lambda(\mu)$ increases as μ increases. If we start with a perturbative value λ_0 at μ_0 and follow the RG flow for increasing μ, then at some scale, $\lambda(\mu)$ may cease to be perturbative. If by a non-perturbative analysis beyond that point one finds $\beta(\lambda) > 0\ \forall\, \lambda$, then $\lambda(\mu)$ increases indefinitely. This can result in a divergent coupling $\lambda \to \infty$, either asymptotically as $\mu \to \infty$, or even for finite values of $\mu \to \mu_\mathrm{L}$. The latter instance is referred to as a **Landau pole**.
   - One appears e.g. in QED at $\mu_\mathrm{L} = \mu_0 \exp\left(\frac{3\pi}{2\alpha_0}\right)$ since

$$\alpha(\mu) = \frac{\alpha_0}{1 - \frac{2\alpha_0}{3\pi}\ln\left(\frac{\mu}{\mu_0}\right)} \xrightarrow{\mu\to\mu_\mathrm{L}} \infty. \tag{24}$$

   On the other hand, $\beta(\lambda) > 0$ means the theory is perturbatively well-defined in the infrared, where λ becomes small. If $\lambda \to 0$ as $\mu \to 0$, the theory even becomes free in the infrared. Such a non-interacting fixed point is called **Gaussian fixed point**.
2. If $\beta(\lambda) < 0$, $\lambda(\mu)$ decreases as μ increases. The theory is perturbative in the UV, but may cease to be perturbative in the IR. If $\lambda \to 0$ as $\mu \to \infty$, the theory becomes free in the UV. This is called **asymptotic freedom**.[4]
3. If $\beta = 0\ \forall \mu$, λ is independent of μ. Such a theory is **conformal**, i.e. scale-independent. Since the counterterms do not induce any scale dependence, there cannot be any UV divergences altogether and the theory is UV finite.

2.4 Wilsonian interpretation

- The original understanding of renormalization was:
  - The cutoff Λ is merely a way to regulate divergent integrals without physical meaning.
  - Renormalization is a trick to remove the cutoff-dependence in physical amplitudes. This procedure allows us to take $\Lambda \to \infty$ without encountering divergences.
  - This comes at the cost of losing predictability for some physical masses and couplings.
  - In a renormalizable theory, only a finite number of such physical couplings must be taken as input parameters from experiment to end up with a well-defined (otherwise predictive) theory.

- The **Wilsonian approach** gives a different interpretation: We should think of QFT as an *effective description* accurate only for energies below an intrinsic cutoff Λ_0. At energies beyond Λ_0 the field theory picture does not correctly model the microscopic degrees of freedom.[5]
  - The only known theory that is UV finite and asymptotes to a weakly coupled QFT in the infrared is **string theory**, which abandons the concept of pointlike particles, replacing them with excitations of a one-dimensional string of length ℓ_s. The string length is the intrinsic cutoff of the low-energy effective QFT. At distances near ℓ_s, the theory deviates from a regular field theory in that it becomes non-local, thus avoiding UV divergences and and arbitrary input parameters.

[4] In $d = 4$, the only known example for asymptotic freedom is Yang-Mills theory.

[5] For example, QFT neglects gravity but all matter gravitates and gravity becomes non-negligible (compared to the other forces) near the Planck scale $M_\mathrm{pl} \approx 1/\sqrt{G_\mathrm{N}} \approx 10^{18}$ GeV.

- Integrating out the degrees of freedom between the regulator Λ and an even smaller cutoff $\Lambda_0 < \Lambda$ yields the **Wilsonian effective action** S_W^{eff}. When computing correlators at scales below Λ_0 via S_W^{eff}, only momenta $|k| \leq \Lambda_0$ appear in the loops since all effects of the modes with $\Lambda_0 < |k| < \Lambda$ are already encoded in S_W^{eff}. This is not to be confused with the quantum effective action $\Gamma[\varphi]$ which gives the full quantum theory already at tree-level. S_W^{eff} includes only those quantum effects due to the integrated-out modes k between $\Lambda_0 < |k| < \Lambda$ and loops must still be performed.
  - Successive applications of this integration to a lower cutoff gives rise to the **renormalisation semi-group** (semi because we can only lower the cutoff; there does not exist an inverse operation).
  - The running couplings in the Wilsonian picture are interpreted as the dependency of the couplings in S_W^{eff} on the cutoff. This identifies Λ_0 as the renormalization scale μ.

3 Quantisation of Yang-Mills theory

3.1 Classical Yang-Mills theory

- The gauge field $A_\mu(x)$ of a Yang-Mills theory with **non-Abelian Lie group** H[6] (whose elements are the gauge transformations that leave the theory invariant) takes values in the associated **Lie algebra** $\mathfrak{h}$. $\mathfrak{h}$ is a vector space equipped with a non-associative antisymmetric bilinear map $[\cdot,\cdot] : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}$, the **Lie bracket**.

- Like any element of $\mathfrak{h}$, $A_\mu(x)$ can be expressed i.t.o. of a basis $\{T^a\}$, $a \in \{1, \dots, \dim(\mathfrak{h})\}$ of $\mathfrak{h}$ (that forms a complete set of generators of the underlying Lie group H):

$$A^\mu(x) = A_a^\mu(x)\, T^a \qquad \text{(summation over } a \text{ implied)} \tag{25}$$

- The basis elements satisfy the Lie algebra's defining relation

$$[T^a, T^b] = i f^{ab}{}_c T^c, \tag{26}$$

  i.t.o. the **structure constants** $f^{ab}{}_c$.[7] Eq. (26) in turn fulfills the **Jacobi identity**

$$\big[[T^a, T^b], T^c\big] + \big[[T^b, T^c], T^a\big] + \big[[T^c, T^a], T^b\big] = 0. \tag{27}$$

  - As an example, the Lie algebra $su(2)$ of dimension 3 has as one possible basis the three Pauli matrices σ_i which generate the corresponding Lie group $SU(2)$. In this basis, the structure constants are given by the components of the Levi-Civita symbol ϵ^{ijk}.

- Every Lie algebra also possesses a symmetric bilinear form, the **Killing form**

$$\kappa^{ab} = T^a \circ T^b, \tag{28}$$

  which is invariant under the **adjoint action** of the Lie group H,

$$h\, T^a\, h^{-1} \circ h\, T^a\, h^{-1} = T^a \circ T^b \qquad \forall\, h \in H. \tag{29}$$

  - When working with H in matrix representation, e.g. $H = SU(N)$, the generators T^a are Hermitian traceless $N \times N$-matrices and the Killing form $\circ$ acts simply as the trace on $\mathfrak{h}$,

$$T^a \circ T^b = \text{tr}_{\mathfrak{h}}(T^a\, T^b) = \frac{1}{2}\, \delta^{ab}. \tag{30}$$

    The last equality only holds if H is compact as a manifold, in which case the Killing form is positive definite and can be suitably normalized.

[6]Generally, any compact, semi-simple Lie group will do, but most forms of Yang-Mills theory are based on $SU(N)$ with associated Lie algebra $\mathfrak{su}(N)$.

[7]The $f^{ab}{}_c$ restrict the result of taking the Lie bracket of two generators T^a, T^b to a linear combination of all generators $\{T^c\}$, thereby determining the Lie brackets of all elements of $\mathfrak{h}$. This almost completely establishes the group structure of H, explaining the name structure constants.

– The Killing form κ^{ab} and its inverse[8] $c\kappa_{ab}$ can be used to raise and lower Lie-algebra indices, e.g.

$$f^{abc} = f^{ab}{}_d \kappa^{dc}. \tag{31}$$

With all indices appearing on the same footing, the structure constants are totally antisymmetric and therefore invariant under cyclic permutations.[9]

- Under a **gauge transformation** $U \in H$ parametrized as $U(x) = e^{-ig\alpha(x)}$ with $g \in \mathbb{R}$ and $\alpha(x) \in \mathfrak{h}$ the gauge potential $A^\mu(x)$ transforms to linear order as

$$A^\mu(x) \to A^\mu(x) + D^\mu \alpha(x), \tag{32}$$

where the **adjoint covariant derivative** acts on $\mathfrak{h}$-valued fields $\alpha(x)$,

$$D^\mu \alpha(x) \equiv \partial^\mu \alpha(x) + ig\big[A^\mu(x), \alpha(x)\big]. \tag{33}$$

- The associated **field strength tensor** of A^μ is given by

$$F_{\mu\nu}(x) = \frac{1}{ig}\big[D_\mu, D_\nu\big] = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + ig\big[A_\mu(x), A_\nu(x)\big]. \tag{34}$$

– $F_{\mu\nu}$ transforms under the adjoint action of H, $F_{\mu\nu} \to U\, F_{\mu\nu} U^{-1}$, and satiefies the Bianchi identity $D_{(\rho} F_{\mu\nu)} = 0$.

– I.t.o. the field strength $F_{\mu\nu}$, the gauge-invariant **Yang-Mills Lagrangian** can be written as

$$\mathcal{L}_{\mathrm{YM}}(A) = -\frac{1}{2}\,\mathrm{tr}_{\mathfrak{h},\mathbb{R}^{1,3}}(F^2) = -\frac{1}{4}\, F^a_{\mu\nu}\, F^{\mu\nu}_a. \tag{35}$$

The commutator in $F_{\mu\nu}$ introduces cubic and quartic gauge field interactions into $\mathcal{L}_{\mathrm{YM}}$. The gauge field's equation of motion is

$$D_\mu F^{\mu\nu} = 0 \qquad \Leftrightarrow \qquad \partial_\mu F^{\mu\nu} = -ig\big[A_\mu(x), F^{\mu\nu}\big]. \tag{36}$$

3.2 Quantizing Yang-Mills theory

- Gauge invariance as well as the fact that $A_0(x)$ appears without a time-derivative in $\mathcal{L}_{\mathrm{YM}}$, i.e. is a non-dynamical field without conjugate momentum Π_0, complicates the quantization of Yang-Mills theory. Variation of the action $S_{\mathrm{YM}}[A]$ w.r.t. to $A_0(x)$ yields $D_i F^{0i} = 0$ ($i \in \{1,2,3\}$) which is itself a **non-dynamical constraint**, with $A_0(x)$ merely an unphysical Lagrange multiplier enforcing it.

- Canonical quantization of constrained systems requires special technology (e.g. the Gupta-Bleuler procedure for $U(1)$ gauge theories). Hence, path integral quantization is preferred for Yang-Mills.

- The naive path integral quantization of a gauge field A^μ proceeds by formulating an action $S[A]$, inverting the kinetic term $(K \cdot A)^\mu = -\partial^2 A^\mu + \partial^\mu \partial_\nu A^\nu$ to find the propagator $iD_F = K^{-1}$, and then perturbatively tackling the interacting theory. This runs into trouble because K is in fact *not* invertible due to its non-trivial kernel $\ker(K) \neq \{0\}$. Rather, $(K \cdot \partial\alpha) = 0\ \forall\, \alpha(x) \in \mathfrak{h}$.[10]

- The cure is to remove the non-invertibility of K by excluding all but one element out of each set of gauge-equivalent field configurations related to first order by $A^\mu \to A^\mu + \partial^\mu \alpha$. Untruncated, a full gauge transformation is given by

$$A^\mu \to A^\mu_h = hA^\mu h^{-1} + \frac{i}{g}(\partial_\mu h)h^{-1}, \qquad \text{for any } h \in H. \tag{37}$$

---

[8] c is merely a normalization factor determined by the structure of the Lie group H as a manifold.

[9] This is not so much naturally the case as up to our choice of basis matrices $\{T^a\}$. It can be proven that a basis always exists in which the f^{abc} have these properties.

[10] This problem is entirely due to gauge invariance and has nothing to do with the gauge group being Abelian or not.

A^μ and A^μ_h are physically equivalent and lead the path integral to overcount because if one satisfies the e.o.m. so does the other. Given any A^μ, all equivalent field configurations lie in the same **orbit**

$$O_A = \{A^\mu_h | h \in H\}. \tag{38}$$

Hence, let $\mathcal{A}$ denote the space of *all* field configurations $A^\mu(x)$, then the physically inequivalent ones are captured precisely by the quotient space $\mathcal{A}/H$ which picks out exactly one field per orbit.

- Some path integral manipulations yield the **Yang-Mills partition function**

$$Z_{\text{YM}} = \int_{\mathcal{A}} \mathcal{D}A \, \delta[F(A)] \, \det(\Delta_{\text{FP}}) \, e^{iS_{\text{YM}}[A]}, \tag{39}$$

where the argument of the functional Dirac delta is the gauge fixing condition $F(A)$ which, given any field configuration $A^\mu(x) \in \mathcal{A}$ achieves $F(A^h) = 0$ (ideally) for exactly one unique $h \in H$[11], thereby effectively reducing the integration domain from $\mathcal{A}$ to $\mathcal{A}/H$. $\Delta_{\text{FP}} = -\frac{\partial F(A)}{\partial A^\mu} D^\mu$ is the **Faddeev-Popov matrix**. Eq. (39) can be used to calculate vacuum expectation values of any (gauge-invariant!) operator $\mathcal{O}(A) = \mathcal{O}(A^h)$ by the usual $t \to \infty(1 - i\epsilon)$ prescription (see eq. (4)).

3.3 Faddeev-Popov ghosts

- Introducing the $\mathfrak{h}$-valued **Nakanishi-Lautrup auxiliary field** $B(x)$, we can rewrite $\delta[F(A)]$ as

$$\delta[F(A)] = \int \mathcal{D}B \, e^{i \int_{\mathbb{R}^{1,3}} \mathrm{d}^4x \, B_a(x) F^a(x)}. \tag{40}$$

With the Grassmann-$\mathfrak{h}$-valued **Faddeev-Popov ghost** $c(x)$ and **antighost** $\bar{c}(x)$, Δ_{FP} becomes

$$\Delta_{\text{FP}} = \int \mathcal{D}\bar{c} \, \mathcal{D}c \, e^{i \int_{\mathbb{R}^{1,3}} \mathrm{d}^4x \, \bar{c}_a(x) \, [\Delta_{\text{FP}} c(x)]^a}. \tag{41}$$

Inserting eqs. (40) and (41) into eq. (39), gives Z_{YM} as

$$Z_{\text{YM}} = \int \mathcal{D}A \, \mathcal{D}B \, \mathcal{D}\bar{c} \, \mathcal{D}c \, e^{iS[A,B,\bar{c},c]}. \tag{42}$$

where the ABc action is

$$S[A, B, \bar{c}, c] = \int_{\mathbb{R}^{1,3}} \mathrm{d}^4x \Big(-\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + B_a(x) F^a(x) + \bar{c}_a(x) \, [\Delta_{\text{FP}} \, c(x)]^a \Big). \tag{43}$$

The ghosts transform as scalar fields under $SO(1,3)$, but have fermionic statistics due to their Grassmannian nature. Thus they violate the spin-statistics theorem as well as unitarity since their Fock space does not have a positive definite norm.

3.4 BRST symmetry and physical Hilbert space[12]

- The problem of determining the physical Hilbert space H_{phys} has two parts. We need to 1. guarantee a positive-definite norm on H_{phys}, and 2. show that time-evolution does not lead out of H_{phys}, i.e. the S-matrix needs to be a unitary operator on H_{phys}. If the criterion for a state to lie in H_{phys} is related to a symmetry of the full interacting theory, then invariance of the physical Hilbert space under time-evolution follows automatically because the S-matrix respects all symmetries.

[11] The ideal case usually doesn't come to pass due to an irritating **residual gauge symmetry** that results in several gauge equivalent field configurations, so-called **Gribov copies**, which all fulfill $F(A) = 0$. Thus even the gauge-fixed path integral would still overcount if we did not restrict it to a fundamental domain where the gauge is unique.

[12] We are finished with the path integral and return to the formalism of canonical quantization from here on.

- Note that the ABc action (43) possesses a **global residual fermionic symmetry** - the **BRST symmetry**. It is implemented via the Grassmann-odd nilpotent operator R acting as

$$R\,A^\mu = -D^\mu c = -(\partial^\mu c + ig[A^\mu, c]), \qquad R\,c = \frac{ig}{2}\{c, \bar{c}\}, \qquad R\,\bar{c} = -B, \qquad R\,B = 0. \tag{44}$$

Defining $\psi(x) = \bar{c}^a \partial_\mu A_a^\mu + \frac{\xi}{2}\bar{c}^a B_a$, with the **gauge-fixing parameter** ξ, the ABc Lagrangian reads

$$\mathcal{L}(A, B, \bar{c}, c) = -\frac{1}{4}\,F^a_{\mu\nu}\,F_a^{\mu\nu} - R\,\psi, \tag{45}$$

which fulfills $R\,\mathcal{L} = 0$, because for the first term, R is just a gauge transformation on A^μ that leaves F^2 invariant, and $R^2 = 0$ in the second.

- Like any quantum symmetry, the **BRST symmetry transformation**

$$\delta_\epsilon \Phi = \epsilon R\,\Phi, \qquad \Phi \in \{A, B, \bar{c}, c\} \tag{46}$$

with ϵ a global Grassmann-valued parameter, is generated (in canonical quantization) by its associated **Noether charge** (operator) $\hat{Q}_{\mathrm{BRST}}$ in the sense that

$$\big[\epsilon \hat{Q}_{\mathrm{BRST}}, \hat{\Phi}\big] = i\delta_\epsilon \hat{\Phi}. \tag{47}$$

The **BRST charge** $\hat{Q}_{\mathrm{BRST}}$ is conserved, nilpotent, and hermitian:

$$\dot{\hat{Q}}_{\mathrm{BRST}} = 0 \;\;\Leftrightarrow\;\; [\hat{H}, \hat{Q}_{\mathrm{BRST}}] = 0, \qquad \hat{Q}^2_{\mathrm{BRST}} = 0, \qquad \hat{Q}^\dagger_{\mathrm{BRST}} = \hat{Q}_{\mathrm{BRST}}. \tag{48}$$

- Some useful mathematics for finding $\mathcal{H}_{\mathrm{phys}}$ of quantum Yang-Mills theory: Any state $|\psi\rangle$ in a vector space $\mathcal{H}$ with a nilpotent linear operator $\hat{Q} : \mathcal{H} \to \mathcal{H}$ acting on it can be classified as either
  - $\hat{Q}$-closed if $\hat{Q}|\psi\rangle = 0$, i.e. if $|\psi\rangle \in \ker(\hat{Q})$, or
  - $\hat{Q}$-exact if $\exists\,|\chi\rangle \in \mathcal{H}$ such that $|\psi\rangle = \hat{Q}|\chi\rangle$, i.e. if $|\psi\rangle \in \mathrm{Im}(\hat{Q})$.

  Since for all $\hat{Q}$-exact $|\psi\rangle$ it holds that $\hat{Q}|\psi\rangle = \hat{Q}^2|\chi\rangle = 0$, we have $\mathrm{Im}(\hat{Q}) \subset \ker(\hat{Q})$. Further, the **$\hat{Q}$-cohomology** is defined as the quotient space $\mathcal{C}(\hat{Q}) \equiv \ker(\hat{Q})/\,\mathrm{Im}(\hat{Q})$.

- Back to physics: The space of states on which the time-evolution operator $\hat{U} = \mathrm{T}e^{i\int_{\mathbb{R}} \hat{H}\,\mathrm{d}t}$ is independent of the specific choice of gauge-fixing condition is given by $\ker(\hat{Q}_{\mathrm{BRST}})$. Time evolution should not depend on gauge, so we require

$$|\psi\rangle \in \mathcal{H}_{\mathrm{phys}} \qquad \text{only if} \quad \hat{Q}_{\mathrm{BRST}}|\psi\rangle = 0. \tag{49}$$

Within $\ker(\hat{Q}_{\mathrm{BRST}})$, the BRST-exact states have zero overlap with all other states since $\forall\,|\psi\rangle \in \mathrm{Im}(\hat{Q}_{\mathrm{BRST}})$ with $|\psi\rangle = \hat{Q}_{\mathrm{BRST}}|\chi\rangle$ and $\forall\,|\xi\rangle \in \ker(\hat{Q}_{\mathrm{BRST}})$

$$\langle\psi|\xi\rangle = \langle\chi|\hat{Q}^\dagger_{\mathrm{BRST}}|\xi\rangle = \langle\chi|\underbrace{\hat{Q}_{\mathrm{BRST}}|\xi\rangle}_{0} = 0. \tag{50}$$

States with zero overlap with all other states can never be measured and are therefore unphysical. We thus define the **physical Hilbert space** of quantum Yang-Mills theory as the cohomology

$$\mathcal{H}_{\mathrm{phys}} = \mathcal{C}(\hat{Q}_{\mathrm{BRST}}) = \frac{\ker(\hat{Q}_{\mathrm{BRST}})}{\mathrm{Im}(\hat{Q}_{\mathrm{BRST}})}. \tag{51}$$

- Evalutating the physical state condition eq. (49) reveals B-,$\bar{c}$-, and c-excitations to be unphysical. The only physical states are of the form

$$|\psi\rangle \in \mathcal{H}_{\mathrm{phys}} \qquad \Leftrightarrow \qquad |\psi\rangle = \xi_\mu |A^\mu(\boldsymbol{k})\rangle \quad \text{with } k^2 = 0 \text{ and } \xi_\mu k^\mu = 0. \tag{52}$$