Quantum Field Theory II – Relevant Exercises

Exam preparation, Advanced QFT, Prof. Jürgen Berges

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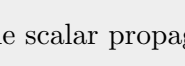
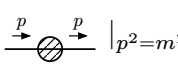
Problem 7.1 (One-loop structure of φ^4 theory)

a) Renormalization of the scalar propagator  $= \frac{i}{p^2 - m^2 - M^2(p^2)}$ up to one-loop order.

Renormalization condition:

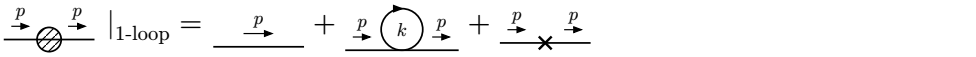


$$= \frac{i}{p^2 - m^2}\,.$$

This requirement is equivalent to

$$\text{i) } M^2(p^2)|_{p^2=m^2} = 0, \qquad \text{ii) } \frac{\mathrm{d}}{\mathrm{d}p^2} M^2(p^2)|_{p^2=m^2} = 0.$$

At one-loop order,  receives contributions from

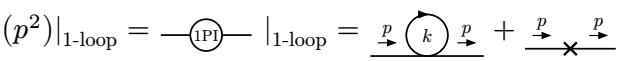


$$= \frac{i}{p^2 - m^2} + \underbrace{\frac{-i\lambda}{2}}_{\text{symmetry factor}} \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} + i(p^2\delta_Z - \delta_m).$$

This follows from the Feynman rules obtained by splitting up the $\lambda\varphi^4$ theory Lagrangian like so

$$\begin{aligned}\mathcal{L}_{\lambda\varphi^4} &= \frac{1}{2}(\partial\varphi_0)^2 - \frac{1}{2}m_0^2\varphi_0^2 - \frac{\lambda_0}{4!}\varphi_0^4 \\ &= \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4!}\varphi^4 + \frac{1}{2}\delta_Z(\partial\varphi)^2 - \frac{1}{2}\delta_m\varphi^2 - \frac{\delta_\lambda}{4!}\varphi^4,\end{aligned}$$

where $\varphi = Z^{-1/2}\varphi_0$, $\delta_Z = Z - 1$, $\delta_m = Zm_0^2 - m^2$, $\delta_\lambda = Z^2\lambda_0 - \lambda$. We know that



thus i) and ii) imply

$$\frac{\mathrm{d}}{\mathrm{d}p^2} M^2(p^2)|_{p^2=m^2} = -\delta_Z = 0,$$

$$M^2(p^2)|_{p^2=m^2} = \frac{\lambda}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2 - m^2} + \delta_m = 0.$$

Thus to complete the renormalization of the propagator of $\lambda\varphi^4$ theory at one-loop order, we simply need to calculate

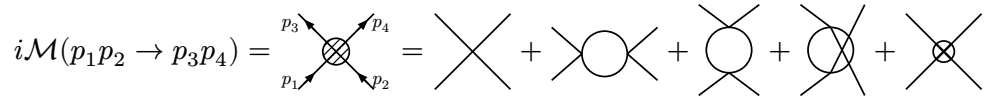
$$\delta_m = -\frac{\lambda}{2}\int \frac{\mathrm{d}^dk}{(2\pi)^d}\frac{i}{k^2 - m^2} \overset{k^0 \to ik_E^0}{=} -\frac{\lambda}{2}\int \frac{\mathrm{d}^dk_E}{(2\pi)^d}\frac{1}{k_E^2 + m^2} = -\frac{\lambda}{2}\int \frac{\mathrm{d}\Omega_d}{(2\pi)^d}\int \mathrm{d}k_E\,\frac{k_E^{d-1}}{k_E^2 + m^2}\,,$$

where $\int \mathrm{d}\Omega_d = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}$. We substitute $x = \frac{m^2}{k_E^2 + m^2}$, $\mathrm{d}x = -\frac{2m^2k_E}{(k_E^2+m^2)^2}\,\mathrm{d}k_E$, $k_E = m\big(\frac{1}{x} - 1\big)^{1/2}$:

$$
\begin{aligned}
\delta_m &= -\frac{2\pi^{d/2}\lambda}{2(2\pi)^d\Gamma\left(\frac{d}{2}\right)}\int\left(-\frac{\left(k_E^2+m^2\right)^2}{2m^2k_E}\,\mathrm{d}x\right)\frac{k_E^{d-1}}{k_E^2+m^2}\\
&= -\frac{\lambda}{2(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\int_0^1 \mathrm{d}x\,\frac{1}{x}\,k_E^{d-2} = -\frac{m^{d-2}\lambda}{2(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\int_0^1 \mathrm{d}x\,\underbrace{\frac{1}{x}\left(\frac{1}{x}-1\right)^{\frac{d}{2}-1}}_{x^{-d/2}(1-x)^{\frac{d}{2}-1}}\\
&= -\frac{m^{d-2}\lambda}{2(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\,\frac{\Gamma\left(1-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(1-\frac{d}{2}+\frac{d}{2}\right)} = -\frac{\lambda m^{d-2}}{2(4\pi)^{d/2}}\underbrace{\Gamma\left(1-\frac{d}{2}\right)}_{\frac{\Gamma(2-d/2)}{1-d/2}}\\
&= -\frac{\lambda m^{2-\varepsilon}}{2(4\pi)^{2-\varepsilon/2}}\,\frac{\Gamma\left(\frac{\varepsilon}{2}\right)}{-\left(1-\frac{\varepsilon}{2}\right)} = \frac{\lambda m^{2-\varepsilon}}{2(4\pi)^{2-\varepsilon/2}}\left(1+\frac{\varepsilon}{2}+\mathcal{O}(\varepsilon^2)\right)\left(\frac{2}{\varepsilon}-\gamma+\mathcal{O}(\varepsilon)\right)\\
&\xrightarrow{\varepsilon\to 0} \frac{m^2\lambda}{2(4\pi)^2}\left(\frac{2}{\varepsilon}-\gamma+1\right) = \frac{m^2\lambda}{32\pi^2}\left(\frac{2}{\varepsilon}-\gamma+1\right).
\end{aligned}
$$

In the end, we found $M^2(p^2)=0\ \forall p$, at least to one-loop order. This is a surprising result and a specialty of φ^4 theory in $d=4$ dimensions.

b) Renormalization of the two-particle scattering amplitude at one loop.

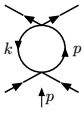
$i\mathcal{M}(p_1p_2\to p_3p_4) =$ 

$$
\begin{aligned}
&= -i\lambda + (-i\lambda)^2\left(iV(s)+iV(t)+iV(u)\right) - i\delta_\lambda.
\end{aligned}
$$

Our renormalization condition is

$$
i\mathcal{M}(p_1p_2\to p_3p_4)\big|_{s=4m^2,t=u=0} = \quad\big|_{s=4m^2,t=u=0} = -i\lambda.
$$

This amounts to $\delta_\lambda = (-i\lambda)^2\left(V(4m^2)+2V(0)\right)$, so all we need to do here is calculate $V(p^2)$, whose structure is irrespective of whether the diagram is s-, t- or u-channel.



$$
= \frac{1}{2}(-i\lambda)^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\,\frac{i}{k^2-m^2}\,\frac{i}{(p+k)^2-m^2} = (-i\lambda)^2 iV(p^2),
$$

$$
V(p^2) = \frac{i}{2}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\,\frac{1}{k^2-m^2}\,\frac{1}{(p+k)^2-m^2} = \frac{i}{2}\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^dk}{(2\pi)^d}\,\frac{1}{\left(x((p+k)^2-m^2)+(1-x)(k^2-m^2)\right)^2},
$$

$$
\begin{aligned}
x((p+k)^2-m^2)+(1-x)(k^2-m^2) &= xp^2+2xpk+x(k^2-m^2)+k^2-m^2-x(k^2-m^2)\\
&= k^2+2xpk+x^2p^2-x^2p^2+xp^2-m^2 = (\underbrace{k+xp}_{l})^2+\underbrace{x(1-x)p^2-m^2}_{\Delta^2},
\end{aligned}
$$

$$
\begin{aligned}
V(p^2) &= \frac{i}{2}\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^dl}{(2\pi)^d}\,\frac{1}{\left(l^2+\Delta^2\right)^2} = -\frac{1}{2}\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^dl_E}{(2\pi)^d}\,\frac{1}{\left(l_E^2-\Delta^2\right)^2} = -\frac{1}{2}\int_0^1\mathrm{d}x\int\frac{\mathrm{d}\Omega_d}{(2\pi)^d}\int_0^\infty\mathrm{d}l_E\,\frac{l_E^{d-1}}{\left(l_E^2-\Delta^2\right)^2}\\
&= -\frac{1}{2}\int_0^1\mathrm{d}x\,\frac{2\pi^{d/2}}{(2\pi)^d\Gamma\left(\frac{d}{2}\right)}\int_1^0\frac{\left(l_E^2-\Delta^2\right)^2}{-2\Delta^2l_E}\,\frac{l_E^{d-1}}{\left(l_E^2-\Delta^2\right)^2}\,\mathrm{d}z,\quad z=\frac{\Delta^2}{l_E^2-\Delta^2},\ l_E=\Delta\left(\frac{1}{z}-1\right)^{1/2},\ \mathrm{d}z=\frac{-2\Delta^2l_E}{\left(l_E^2-\Delta^2\right)^2}\,\mathrm{d}l_E\\
&= -\frac{1}{2(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\int_0^1\mathrm{d}x\,\underbrace{\int_0^1\mathrm{d}z\,z^{1-\frac{d}{2}}(1-z)^{\frac{d}{2}-1}}_{\Gamma(2-d/2)\Gamma(d/2)/\Gamma(2-d/2+d/2)}\,\Delta^{d-4} = -\frac{\Gamma\left(2-\frac{d}{2}\right)}{2(4\pi)^{d/2}}\int_0^1\mathrm{d}x\,\Delta^{d-4}.
\end{aligned}
$$

Using $\Gamma\big(2-\frac{d}{2}\big)=\Gamma\big(\frac{\varepsilon}{2}\big)=\frac{2}{\varepsilon}-\gamma+\mathcal{O}(\varepsilon)$, $X^{-\varepsilon/2}=e^{\ln X^{-\varepsilon/2}}=e^{-\frac{\varepsilon}{2}\ln X}=1-\frac{\varepsilon}{2}\ln X+\mathcal{O}(\varepsilon^2)$, we get

$$\begin{aligned}V(p^2)&=-\frac{\Gamma\big(\frac{\varepsilon}{2}\big)}{2(4\pi)^2}(4\pi)^{\varepsilon/2}\int_0^1\mathrm{d}x\,\big(x(1-x)p^2-m^2\big)^{-\varepsilon/2}\\&=-\frac{1}{32\pi^2}\left(\frac{2}{\varepsilon}-\gamma+\mathcal{O}(\varepsilon)\right)\left(1+\frac{\varepsilon}{2}\ln(4\pi)+\mathcal{O}(\varepsilon^2)\right)\int_0^1\mathrm{d}x\left[1-\frac{\varepsilon}{2}\ln(x(1-x)p^2-m^2)+\mathcal{O}(\varepsilon^2)\right]\\&=-\frac{1}{32\pi^2}\int_0^1\mathrm{d}x\left[\frac{2}{\varepsilon}-\gamma+\ln(4\pi)-\ln(x(1-x)p^2-m^2)+\mathcal{O}(\varepsilon)\right].\end{aligned}$$

So the divergent counterterm δ_λ is given by

$$\delta_\lambda=(-i\lambda)^2(V(4m^2)+2V(0))=\frac{\lambda^2}{32\pi^2}\int_0^1\mathrm{d}x\left[\frac{6}{\varepsilon}-3\gamma+3\ln(4\pi)-\ln(x(1-x)4m^2-m^2)-2\ln(m^2)\right]$$

and for the two-particle scattering amplitude, we obtain the lengthy but finite result of

$$i\mathcal{M}(p_1p_2\to p_3p_4)=-i\lambda-\frac{i\lambda^2}{32\pi^2}\int_0^1\mathrm{d}x\left[\ln\left(\frac{x(1-x)s-m^2}{x(1-x)4m^2-m^2}\right)+\ln\left(\frac{x(1-x)t-m^2}{m^2}\right)+\ln\left(\frac{x(1-x)u-m^2}{m^2}\right)\right].$$

Problem 8.1 (Superficial degree of divergence)

Aim: derivation of the superficial degree of divergence D of QED.

Notation:
- $E_{f/\gamma}$: number of external fermion/photon lines
- $P_{f/\gamma}$: number of internal fermion/photon lines (propagators)
- V: number of vertices
- L: number of loops

In $d=4$ dimensions, every loop gives a four-momentum integral. Every fermionic/photonic propagator gives one/two inverse powers of momentum. Thus,

$$D=4L-P_f-2P_\gamma.$$

Every propagator connects to two vertices while every external line connects to one vertex, since we are concerned solely with fully connected amputated diagrams, i.e. no external lines simply going through and no propagators on external lines. Further, every vertex connects two fermionic lines to one photon, so

$$V=\frac{1}{2}\big(E_f+2P_f\big)=E_\gamma+2P_\gamma,$$

and each propagator yields an integral over its undetermined momentum while every vertex gives a delta distribution, determining one momentum. Of these delta distributions, however, one is used up to assure overall momentum conservation. So the number of momentum integrals (equals number of loops) is

$$L=P_f+P_\gamma-(V-1)=P_f+P_\gamma-V+1.$$

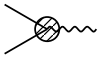
Putting everything together, we can construct an expression for D which depends only on the external lines:

$$\begin{aligned}D&=4L-P_f-2P_\gamma=4P_f+4P_\gamma-4V+4-P_f-2P_\gamma\\&=4+3P_f+2P_\gamma-3V-V\end{aligned}$$

$$= 4 + 3P_f + 2P_\gamma - \frac{3}{2}\big(E_f + 2P_f\big) - E_\gamma - 2P_\gamma$$
$$= 4 - \frac{3}{2}E_f - E_\gamma.$$

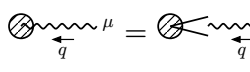
Problem 8.2 (One-loop structure of QED)

a) We are to give D for the following seven amplitudes.

i) vacuum energy		$E_f = E_\gamma = 0,\ D = 4$
ii) photon propagator		$E_f = 0,\ E_\gamma = 2,\ D = 2$
iii) electron propagator		$E_f = 2,\ E_\gamma = 0,\ D = 1$
iv) electron-electron-photon vertex		$E_f = 2,\ E_\gamma = 1,\ D = 0$
v) 1-photon amplitude		$E_f = 0,\ E_\gamma = 1,\ D = 3$
vi) 3-photon amplitude		$E_f = 0,\ E_\gamma = 3,\ D = 1$
vii) 4-photon amplitude		$E_f = 0,\ E_\gamma = 4,\ D = 0$

b) Aim: Show that the one-loop photon one-point function vanishes by the discrete symmetry under charge conjugation.

The only way for an external photon to enter a diagram is via a QED vertex, so we may 'drag' that out of and write



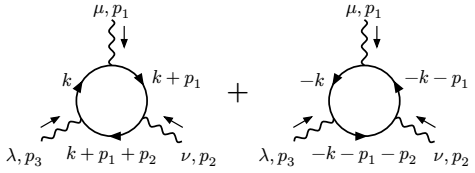
$$= -ie \int \mathrm{d}^4x\, e^{-iqx} \langle \Omega \mid T j^\mu(x) \mid \Omega \rangle,$$

with $j^\mu(x) = \overline{\Psi}(x)\gamma^\mu\Psi(x)$. Since $\hat{C}$ is a symmetry of QED, i.e. $\hat{C}\,|\Omega\rangle = |\Omega\rangle$, but changes the sign of any current $j^\mu(x)$, the amplitude must be zero:

$$\langle \Omega \mid T j^\mu(x) \mid \Omega \rangle = \langle \Omega \mid \underbrace{\hat{C}^2}_{\mathbb{1}} T j^\mu(x) \underbrace{\hat{C}^2}_{\mathbb{1}} \mid \Omega \rangle = \langle \Omega \mid \hat{C} T \underbrace{\hat{C} j^\mu(x) \hat{C}}_{-j^\mu(x)} \hat{C} \mid \Omega \rangle$$
$$= -\langle \Omega \mid T j^\mu(x) \mid \Omega \rangle = 0.$$

Aim: Show that the two diagrams contributing to the photon three-point function at one-loop order cancel.

The contributions to $\Gamma^{(3)}$ at one-loop order are given by fermionic loops that differ only in their direction of momentum flow. Using the fermion propagator $S_F(k) = \frac{i}{\not{k} - m}$, we write

$$\Gamma^{(3)}\big|_{\text{1-loop}} =$$


$$= (-ie)^3 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \big[-\operatorname{tr}\big(S_F(k)\gamma^\mu S_F(k+p_1)\gamma^\nu S_F(k+p_1+p_2)\gamma^\lambda\big)$$
$$- \operatorname{tr}\big(S_F(-k)\gamma^\mu S_F(-k-p_1)\gamma^\nu S_F(-k-p_1-p_2)\gamma^\lambda\big)\big].$$

The only thing in a fermionic propagator $S_F(k)$ affected by charge conjugation is the γ-matrix. By requiring that charge conjugation of the Dirac equation for a field Ψ with charge e gives the Dirac equation for a field Ψ^c with charge $-e$, we obtain the condition $\hat{C}\gamma^\mu\hat{C} = -(\gamma^\mu)^T$. Thus

$$\hat{C}S_F(-k)\hat{C} = \hat{C}\,\frac{i}{-\not{k}-m}\,\hat{C} = \hat{C}\,\frac{i(-\not{k}+m)}{k^2-m^2}\,\hat{C} = \frac{i\big((\gamma^\mu)^T k_\mu + m\big)}{k^2-m^2} = \frac{i(\not{k}^T+m)}{k^2-m^2} = S_F^T(k).$$

Applying this to $i\Gamma^{(3)}|_{\text{1-loop}}$, we find

$$\begin{aligned} i\Gamma^{(3)}|_{\text{1-loop}} &= (-ie)^3 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \Big[-\operatorname{tr}\big(S_F(k)\gamma^\mu S_F(k+p_1)\gamma^\nu S_F(k+p_1+p_2)\gamma^\lambda\big) \\ &\qquad - \operatorname{tr}\big(\hat{C}^2 S_F(-k)\hat{C}^2\gamma^\mu\hat{C}^2 S_F(-k-p_1)\hat{C}^2\gamma^\nu\hat{C}^2 S_F(-k-p_1-p_2)\hat{C}^2\gamma^\lambda\big)\Big] \\ &= (-ie)^3 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \Big[-\operatorname{tr}\big(S_F(k)\gamma^\mu S_F(k+p_1)\gamma^\nu S_F(k+p_1+p_2)\gamma^\lambda\big) \\ &\qquad + \operatorname{tr}\Big(S_F^T(k)(\gamma^\mu)^T S_F^T(k+p_1)(\gamma^\nu)^T S_F^T(k+p_1+p_2)\big(\gamma^\lambda\big)^T\Big)\Big] = 0. \end{aligned}$$

Aim: Show that the diagrams contributing to any n-point photon amplitude, for n odd, cancel in pairs.

Obviously, we may employ the above scheme for any fermionic loop with an odd number of vertices.

Problem 9.1 (Renormalization of Yukawa theory)

$$\mathcal{L}_Y = \frac{1}{2}(\partial\varphi_0)^2 - \frac{1}{2}m_0^2\varphi_0^2 + \overline{\Psi}_0(i\not{\partial} - M_0)\Psi_0 - ig_0\varphi_0\overline{\Psi}_0\gamma^5\Psi_0$$

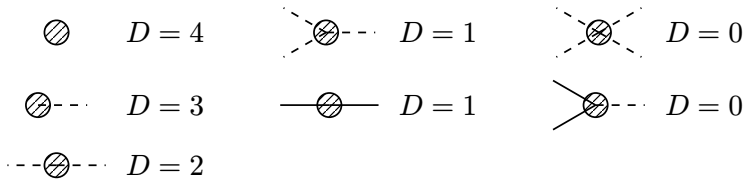
is invariant under parity $\Psi(t,\vec{x}) \to \gamma^0\Psi(t,-\vec{x})$ and $\varphi(t,\vec{x}) \to -\varphi(t,-\vec{x})$.

a) Determine the superficially divergent amplitudes and work out the Feynman rules for renormalized perturbation theory.

The superficial degree of divergence is the same as that of QED since the Lagrangian has the same structure; we only need to replace the gauge field A_μ with the scalar field φ:

$$\left.\begin{aligned} D &= 4L - P_f - 2P_\varphi \\ L &= P_f + P_\varphi - V + 1 \\ V &= \tfrac{1}{2}\big(E_f + 2P_f\big) = E_\varphi + 2P_\varphi \end{aligned}\right\} \implies D = 4 - \frac{3}{2}E_f - E_\varphi.$$

Superficially divergent amplitudes are those with $D \geq 0$:



To obtain the Feynman rules, we split up $\mathcal{L} = \mathcal{L}_Y + \delta\mathcal{L}$, $\delta\mathcal{L} = -\frac{\lambda_0}{4!}\varphi_0^4$,

$$\begin{aligned} \mathcal{L} = {} & \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 + \overline{\Psi}(i\not{\partial} - M)\Psi - ig\varphi\overline{\Psi}\gamma^5\Psi - \frac{\lambda}{4!}\varphi^4 \\ & + \frac{1}{2}\delta_\varphi(\partial\varphi)^2 - \frac{1}{2}\delta_m\varphi^2 + \overline{\Psi}(i\delta_\Psi\not{\partial} - \delta_M)\Psi - i\delta_g\varphi\overline{\Psi}\gamma^5\Psi - \frac{\delta_\lambda}{4!}\varphi^4 \end{aligned}$$

into renormalized fields, $\varphi = Z_\varphi^{-1/2}\varphi_0$, and parameters m, M, g, and the counterterms,

$\delta_\varphi = Z_\varphi - 1, \quad \delta_m = m_0^2 Z_\varphi - m^2, \quad \delta_\Psi = Z_\Psi - 1, \quad \delta_M = M_0 Z_\Psi - M, \quad \delta_g = g_0 Z_\varphi^{1/2} Z_\Psi - g, \quad \delta_\lambda = \lambda_0 Z_\varphi^2 - \lambda$:

$$\tfrac{1}{2}(\partial\varphi)^2 - \tfrac{1}{2}m^2\varphi^2 \mathrel{\hat{=}} \quad \overset{k}{\rightarrow} \quad = \tfrac{i}{k^2-m^2}$$

$$\overline{\Psi}(i\not{\partial} - M)\Psi \mathrel{\hat{=}} \quad \overset{p}{\rightarrow} \quad = \tfrac{i}{\not{p}-M} = \tfrac{i(\not{p}+M)}{p^2-M^2}$$

$$-ig\varphi\overline{\Psi}\gamma^5\Psi \mathrel{\hat{=}} \quad = g\gamma^5,$$

$$-\tfrac{\lambda}{4!}\varphi^4 \mathrel{\hat{=}} \quad = -i\lambda$$

$$\tfrac{1}{2}\delta_\varphi(\partial\varphi)^2 - \tfrac{1}{2}\delta_m\varphi^2 \mathrel{\hat{=}} \quad = i\big(p^2\delta_\varphi - \delta_m\big)$$

$$\overline{\Psi}(i\delta_\Psi\not{\partial} - \delta_M)\Psi \mathrel{\hat{=}} \quad = i(\not{p}\delta_\Psi - \delta_M)$$

$$-i\delta_g\varphi\overline{\Psi}\gamma^5\Psi \mathrel{\hat{=}} \quad = \delta_g\gamma^5$$

$$-\tfrac{\delta_\lambda}{4!}\varphi^4 \mathrel{\hat{=}} \quad = -i\delta_\lambda$$

b) Aim: compute the divergent part of each counterterm to one-loop order.

- ⊘ no computation necessary, can be trivially absorbed into the vacuum energy constant V_0 in the Lagrangian.
- ⊘--- $= 0$ by parity invariance of the Lagrangian.

$$\text{--}\oslash\text{--}\,|_{\text{1-loop}} = \text{------} + \overset{k}{\text{---}} + \overset{p+k}{\text{--}}\overset{p}{\text{--}}\overset{k}{\text{--}}\overset{p}{\text{--}} + \text{--}\times\text{--}$$

$$= \text{------} + \overbrace{iM^2(p^2)|_{\text{1-loop}}}^{=0 \text{ at } p^2=m^2},$$

$$\overset{k}{\text{---}} = -\frac{i}{2}\lambda\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2} = \frac{im^2\lambda}{32\pi^2}\Big(\frac{2}{\varepsilon} - \gamma + 1 + \ln(4\pi)\Big),$$

$$\overset{p+k}{\underset{p}{\text{--}}}\overset{p}{\text{--}}\underset{k}{\text{--}} = -g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{tr}\Big(\gamma^5\frac{i(\not{k}+M)}{k^2-M^2}\gamma^5\frac{i(\not{k}+\not{p}+M)}{(k+p)^2-M^2}\Big) = g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{\mathrm{tr}(\gamma^5(\not{k}+M)\gamma^5(\not{k}+\not{p}+M))}{(k^2-M^2)((k+p)^2-M^2)},$$

$$\mathrm{tr}(\gamma^5(\not{k}+M)\gamma^5(\not{k}+\not{p}+M)) = \mathrm{tr}(-\not{k}^2 - \not{k}\not{p} + \not{p}M + M^2) = 4(M^2 - k^2 - k\cdot p).$$

For the denominator, we use a Feynman parametrization,

$$(k^2-M^2)^{-1}((k+p)^2-M^2)^{-1} = \int_0^1 \mathrm{d}x\,\frac{1}{\big[x((k+p)^2-M^2) + (1-x)(k^2-M^2)\big]^2},$$

$$x((k+p)^2-M^2) + (1-x)(k^2-M^2) = xp^2 + 2xkp + k^2 - M^2 = \underbrace{(k+xp)}_{l}{}^2 \underbrace{+x(1-x)p^2 - M^2}_{-\Delta},$$

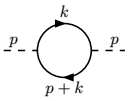
$$\overset{k}{\underset{p+k}{\text{--}}}\overset{p}{\text{--}} = -4g^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\int_0^1\mathrm{d}x\,\frac{k^2-M^2+kp}{(l^2-\Delta)^2} = -4g^2\int\frac{\mathrm{d}^dl}{(2\pi)^d}\int_0^1\mathrm{d}x\,\frac{l^2 - \overbrace{2xlp}^{0} + x^2p^2 - M^2 + \overbrace{lp}^{0} - xp^2}{(l^2-\Delta)^2}$$

$$= -4g^2\int\frac{\mathrm{d}^dl}{(2\pi)^d}\int_0^1\mathrm{d}x\,\frac{l^2 - x(1-x)p^2 - M^2}{(l^2-\Delta)^2}.$$

Using the formulae of dimensional regularization,

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d}\,\frac{1}{(l^2-\Delta)^n} = \frac{i(-1)^n}{(4\pi)^{d/2}}\,\frac{\Gamma\big(n-\frac{d}{2}\big)}{\Gamma(n)}\,\Delta^{\frac{d}{2}-n},$$
$$\int \frac{\mathrm{d}^d l}{(2\pi)^d}\,\frac{l^2}{(l^2-\Delta)^n} = \frac{i(-1)^{n-1}}{(4\pi)^{d/2}}\,\frac{d}{2}\,\frac{\Gamma\big(n-\frac{d}{2}-1\big)}{\Gamma(n)}\,\Delta^{1+\frac{d}{2}-n},$$

we get



$$\begin{aligned}
&= -4g^2\int_0^1 \mathrm{d}x\left(\frac{-i\,\frac{d}{2}\,\Gamma\big(1-\frac{d}{2}\big)}{(4\pi)^{d/2}}\,\Delta^{\frac{d}{2}-1} - \frac{i\Gamma\big(2-\frac{d}{2}\big)}{(4\pi)^{d/2}}\,\Delta^{\frac{d}{2}-2}(x(1-x)p^2+M^2)\right)\\
&= -\frac{4ig^2}{(4\pi)^{d/2}}\int_0^1 \mathrm{d}x\,\Gamma\left(2-\frac{d}{2}\right)\Delta^{\frac{d}{2}-2}\left(\frac{\frac{d}{2}}{\frac{d}{2}-1}\,\Delta - x(1-x)p^2 - M^2\right)\\
&\xrightarrow{\varepsilon\to 0} -\frac{8ig^2}{16\pi^2\varepsilon}\int_0^1 \mathrm{d}x\,(2\Delta - x(1-x)p^2 - M^2) + \text{finite} = -\frac{8ig^2}{16\pi^2\varepsilon}\int_0^1 \mathrm{d}x\,(M^2 - 3x(1-x)p^2) + \text{finite}\\
&= \frac{ig^2}{4\pi^2}\,\frac{p^2-2M^2}{\varepsilon} + \text{finite terms.}
\end{aligned}$$

Problem 10.1 (One-loop β-function in QED)

a) Aim: compute the one-loop β-function in QED.

$$\beta_e(\mu) = \mu\frac{\partial}{\partial\mu}\left(-e\delta_1 + e\delta_2 + \frac{1}{2}e\delta_3\right),$$
$$\begin{aligned}
\delta_1 &= -\frac{e^2}{(4\pi)^{d/2}}\int_0^1 \mathrm{d}z(1-z)\left[\frac{\Gamma\big(2-\frac{d}{2}\big)}{\left[(1-z)^2m^2+zM^2\right]^{2-d/2}}\,\frac{(2-\varepsilon)^2}{2}\right.\\
&\qquad\left. + \frac{\Gamma\big(3-\frac{d}{2}\big)}{\left[(1-z)^2m^2+zM^2\right]^{3-d/2}}\,(2(1-4z+z^2)-\varepsilon(1-z)^2)m^2\right],\\
\delta_2 &= -\frac{e^2}{(4\pi)^{d/2}}\int_0^1 \mathrm{d}z\,\frac{\Gamma\big(2-\frac{d}{2}\big)}{\left[(1-z)^2m^2+zM^2\right]^{2-d/2}}\left[(2-\varepsilon)z - \frac{\varepsilon}{2}\,\frac{2z(1-z)m^2}{(1-z)^2m^2+zM^2}\,(4-2z-\varepsilon(1-z))\right],\\
\delta_3 &= -\frac{e^2}{(4\pi)^{d/2}}\int_0^1 \mathrm{d}z\,\frac{\Gamma\big(2-\frac{d}{2}\big)}{(m^2)^{2-d/2}}\,(8z(1-z)),
\end{aligned}$$

where M is an IR cutoff and m is the electron mass, playing the role of the renormalization scale μ. Since M is just an IR cutoff and QED is well-defined in the IR, we can, right from the outset, set it to zero. Thus

$$
\begin{aligned}
m\frac{\partial\delta_1}{\partial m} &= m\frac{\partial}{\partial m}\left[-\frac{e^2}{(4\pi)^{d/2}}\int_0^1 \mathrm{d}z(1-z)\left[\frac{\Gamma(\frac{\varepsilon}{2})}{((1-z)m)^\varepsilon}\,\frac{(2-\varepsilon)^2}{2}+\frac{\Gamma(1+\frac{\varepsilon}{2})}{(1-z)^{2+\varepsilon}m^\varepsilon}\,(2(1-4z+z^2)-\varepsilon(1-z)^2)\right]\right]\\
&= \frac{\varepsilon e^2}{(4\pi)^{d/2}m^\varepsilon}\int_0^1 \mathrm{d}z\left[\frac{\frac{2}{\varepsilon}-\gamma+\mathcal{O}(\varepsilon)}{2(1-z)^{\varepsilon-1}}\,(4-4\varepsilon+\varepsilon^2)+\frac{\frac{\varepsilon}{2}\left(\frac{2}{\varepsilon}-\gamma+\mathcal{O}(\varepsilon)\right)}{(1-z)^{1+\varepsilon}}\,(2(1-4z+z^2)-\varepsilon(1-z)^2)\right]\\
&\xrightarrow{\varepsilon\to0} \frac{e^2}{(4\pi)^2}\int_0^1 \mathrm{d}z\,4(1-z)=\frac{e^2}{8\pi^2},\\
m\frac{\partial\delta_2}{\partial m} &= m\frac{\partial}{\partial m}\left(-\frac{e^2}{(4\pi)^{d/2}}\int_0^1 \mathrm{d}z\,\frac{\Gamma(\frac{\varepsilon}{2})}{((1-z)m)^\varepsilon}\left[(2-\varepsilon)z-\frac{\varepsilon}{2}2z(4-2z-\varepsilon(1-z))\right]\right)\\
&= \frac{\varepsilon e^2}{(4\pi)^{d/2}m^\varepsilon}\int_0^1 \mathrm{d}z\,\frac{\frac{2}{\varepsilon}-\gamma+\mathcal{O}(\varepsilon)}{(1-z)^\varepsilon}\left[(2-\varepsilon)z-\varepsilon(4-2z-\varepsilon(1-z))\right]\\
&\xrightarrow{\varepsilon\to0} \frac{e^2}{(4\pi)^2}\int_0^1 \mathrm{d}z\,2=\frac{e^2}{8\pi^2},\\
m\frac{\partial\delta_3}{\partial m} &= m\frac{\partial}{\partial m}\left(-\frac{e^2}{(4\pi)^{d/2}}\int_0^1 \mathrm{d}z\,\frac{\Gamma(\frac{\varepsilon}{2})}{m^\varepsilon}8z(1-z)\right)=\frac{\varepsilon e^2}{(4\pi)^{d/2}m^\varepsilon}\int_0^1 \mathrm{d}z\left(\frac{2}{\varepsilon}-\gamma+\mathcal{O}(\varepsilon)\right)8z(1-z)\\
&\xrightarrow{\varepsilon\to0} \frac{e^2}{(4\pi)^2}\int_0^1 \mathrm{d}z\,16z(1-z)=\frac{e^2}{6\pi^2}.
\end{aligned}
$$

Inserting δ_1, δ_2, δ_3 into the one-loop β-function of QED, we get

$$
\beta_e(m) = -e\frac{e^2}{8\pi^2}+e\frac{e^2}{8\pi^2}+\frac{1}{2}e\frac{e^2}{6\pi^2}=\frac{e^3}{12\pi^2}.
$$

b) Aim: Integrate the RG equation to find the running of the coupling.

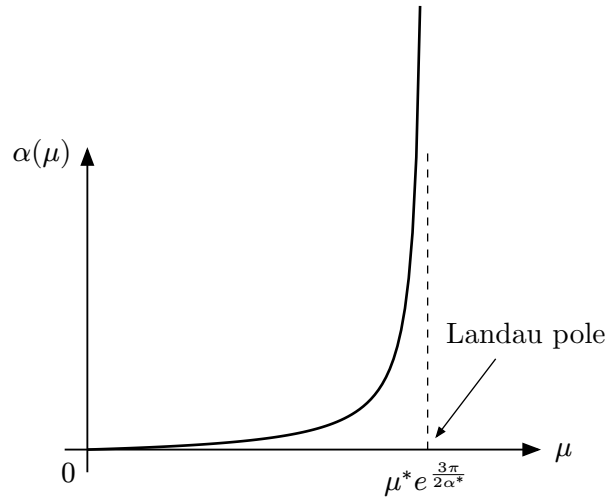
RG equation:

$$
\frac{\mathrm{d}}{\mathrm{d}\ln\mu}e(\mu)=\beta_e(\mu)=\frac{e^3(\mu)}{12\pi^2}\quad\Longrightarrow\quad 12\pi^2\frac{\mathrm{d}e(\mu)}{e^3(\mu)}=\mathrm{d}\ln\mu,
$$

$$
\begin{aligned}
\int_{\mu^*}^{\mu}12\pi^2\frac{\mathrm{d}e(\mu')}{e^3(\mu')} &\overset{(*)}{=} \int_{\mu^*}^{\mu}12\pi^2\frac{2\pi}{e(\mu')}\,\mathrm{d}\alpha(\mu')\frac{1}{e^3(\mu')}=\frac{3}{2}\pi\int_{\mu^*}^{\mu}\frac{\mathrm{d}\alpha(\mu')}{\alpha^2(\mu')}=-\frac{3}{2}\pi\left(\frac{1}{\alpha(\mu)}-\underbrace{\frac{1}{\alpha(\mu^*)}}_{=:\alpha^*}\right)\\
&= \int_{\mu^*}^{\mu}\mathrm{d}\ln\mu'=\ln\left(\frac{\mu}{\mu^*}\right)\quad\Longrightarrow\quad\frac{1}{\alpha(\mu)}-\frac{1}{\alpha^*}=-\frac{2}{3\pi}\ln\left(\frac{\mu}{\mu^*}\right),
\end{aligned}
$$

where in $(*)$ we substituted $\alpha(\mu)=e^2(\mu)/(4\pi)$, $\mathrm{d}\alpha(\mu)=e(\mu)/(2\pi)\,\mathrm{d}e(\mu)$.

$$
\alpha(\mu)=\frac{1}{\frac{1}{\alpha^*}-\frac{2}{3\pi}\ln\left(\frac{\mu}{\mu^*}\right)}=\frac{\alpha^*}{1-\frac{2\alpha^*}{3\pi}\ln\left(\frac{\mu}{\mu^*}\right)},\quad\text{pole at }1=\frac{2\alpha^*}{3\pi}\ln\left(\frac{\mu}{\mu^*}\right)\Longrightarrow\mu=\mu^*e^{\frac{3\pi}{2\alpha^*}}.
$$



$e(\mu)$ ceases to be perturbative if it is of the order $e(\mu) \approx 1$, i.e.

$$\frac{1}{4\pi} = \frac{1}{\frac{1}{\alpha^*} - \frac{2}{3\pi}\ln\left(\frac{\mu}{\mu^*}\right)} \quad \Longrightarrow \quad \ln\left(\frac{\mu}{\mu^*}\right) = \frac{3\pi}{2}\left(\frac{1}{\alpha^*} - 4\pi\right), \quad \mu = \mu^* e^{\frac{3\pi}{2}\left(\frac{1}{\alpha^*} - 4\pi\right)}.$$

Problem 11.2 (Fermion-antifermion annihilation in Yang–Mills theory at tree level)

a) Aim: Use the Feynman rules to compute the t- and u-channel of this process.

$$i\mathcal{M}_1^{ab} = \quad = \overline{v}(p_1)(ig\gamma^\mu T^a)\frac{i(\not{q}+m)}{q^2-m^2}\left(ig\gamma^\nu T^b\right)u(p_2)\varepsilon_\mu^*(k_1)\varepsilon_\nu^*(k_2),$$

$$i\mathcal{M}_2^{ab} = \quad = \overline{v}(p_1)\left(ig\gamma^\nu T^b\right)\frac{i(\not{q}+m)}{q^2-m^2}(ig\gamma^\mu T^a)u(p_2)\varepsilon_\mu^*(k_1)\varepsilon_\nu^*(k_2),$$

where the internal fermion carries $q = -(p_1 - k_1) = p_2 - k_2$ in the first (t-channel) and $q = -(p_1 - k_2) = p_2 - k_1$ in the second (u-channel) diagram.

$$i\mathcal{M}_{1+2}^{ab} = i\mathcal{M}_1^{ab} + i\mathcal{M}_2^{ab} = -i(ig)^2\overline{v}(p_1)\left[\gamma^\mu T^a\frac{\not{p}_1 - \not{k}_1 + m}{(p_1-k_1)^2 - m^2}T^b\gamma^\nu + \gamma^\nu T^b\frac{\not{p}_1 - \not{k}_2 + m}{(p_1-k_2)^2 - m^2}T^a\gamma^\mu\right]u(p_2)\varepsilon_\mu^*(k_1)\varepsilon_\nu^*(k_2).$$

If $\varepsilon_\nu^*(k_2) = k_{2\nu}$, we may use the Dirac equation to intelligently add zero:

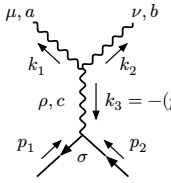
$$\begin{aligned}
&\not{k}_2 u(p_2) = -(\not{p}_2 - \not{k}_2 - m)u(p_2) = (\not{p}_1 - \not{k}_1 + m)u(p_2), && \text{since} \quad -(\not{p}_2 - m)u(p_2) = 0,\\
&\overline{v}(p_1)\not{k}_2 = -\overline{v}(p_1)(\not{p}_1 - \not{k}_2 + m), && \text{since} \quad -\overline{v}(p_1)(\not{p}_1 + m) = 0.
\end{aligned}$$

Thus, for $\varepsilon_\nu^*(k_2) = k_{2\nu}$, $i\mathcal{M}_{1+2}^{ab}$ becomes

$$i\mathcal{M}_{1+2}^{ab} = -i(ig)^2\overline{v}(p_1)\left[\gamma^\mu T^a\frac{\not{p}_1 - \not{k}_1 + m}{(p_1-k_1)^2 - m^2}T^b\not{k}_2 + \not{k}_2 T^b\frac{\not{p}_1 - \not{k}_2 + m}{(p_1-k_2)^2 - m^2}T^a\gamma^\mu\right]u(p_2)\varepsilon_\mu^*(k_1)$$

$$= -i(ig)^2\overline{v}(p_1)\left[\gamma^\mu T^a\frac{(p_1-k_1)^2-m^2}{(p_1-k_1)^2-m^2}T^b - T^b\frac{(p_1-k_2)^2-m^2}{(p_1-k_2)^2-m^2}T^a\gamma^\mu\right]u(p_2)\varepsilon^*_\mu(k_1)$$
$$= (ig)^2\overline{v}(p_1)[-i\gamma^\mu[T^a,T^b]]u(p_2)\varepsilon^*_\mu(k_1).$$

b) Aim: Use the Feynman rules to compute the s-channel.

$i\mathcal{M}_3^{ab} =$ 

$$= \overline{v}(p_1)(ig\gamma^\sigma T^c)u(p_2)\frac{-i\eta_{\rho\sigma}}{k_3^2}gf^{abc}[\eta^{\mu\nu}(k_1-k_2)^\rho + \eta^{\nu\rho}(k_2-k_3)^\mu + \eta^{\rho\mu}(k_3-k_1)^\nu]\varepsilon^*_\mu(k_1)\varepsilon^*_\nu(k_2)$$
$$= g^2\overline{v}(p_1)\gamma_\rho u(p_2)\frac{1}{k_3^2}f^{abc}T^c[\eta^{\mu\nu}(k_1-k_2)^\rho + \eta^{\nu\rho}(k_2-k_3)^\mu + \eta^{\rho\mu}(k_3-k_1)^\nu]\varepsilon^*_\mu(k_1)\varepsilon^*_\nu(k_2).$$

Again assuming longitudinal polarization, $\varepsilon^*_\nu(k_2) = k_{2\nu}$, for one outgoing photon and transverse polarization, $\varepsilon^*_\mu(k_1)k_1^\mu = 0$, for the other, we get

$$i\mathcal{M}_3^{ab} = g^2\overline{v}(p_1)\gamma_\rho u(p_2)\frac{1}{k_3^2}f^{abc}T^c[k_2^\mu(k_1-k_2)^\rho + k_2^\rho(k_2-k_3)^\mu + \eta^{\mu\rho}(k_3-k_1)\cdot k_2]\varepsilon^*_\mu(k_1),$$

$$[k_2^\mu(k_1-k_2)^\rho + k_2^\rho(k_2-k_3)^\mu + \eta^{\mu\rho}(k_3-k_1)\cdot k_2]\varepsilon^*_\mu(k_1)$$
$$= [-(k_1+k_3)^\mu(2k_1+k_3)^\rho + (k_1+k_3)^\rho(k_1+2k_3)^\mu - \eta^{\mu\rho}(k_3-k_1)\cdot(k_1+k_3)]\varepsilon^*_\mu(k_1)$$
$$\overset{k_2=-k_1-k_3}{=} [-2k_1^\mu k_1^\rho - \cancel{k_1^\mu k_3^\rho} - \cancel{2k_1^\rho k_3^\mu} - k_3^\mu k_3^\rho + k_1^\mu k_1^\rho + \cancel{2k_1^\rho k_3^\mu} + \cancel{k_1^\mu k_3^\rho} + 2k_3^\mu k_3^\rho + \eta^{\mu\rho}(\underbrace{k_1^2}_{0} - k_3^2)]\varepsilon^*_\mu(k_1)$$
$$= [-k_1^\mu k_1^\rho + k_3^\mu k_3^\rho - \eta^{\mu\rho}k_3^2]\varepsilon^*_\mu(k_1) = (k_3^\mu k_3^\rho - \eta^{\mu\rho}k_3^2)\varepsilon^*_\mu(k_1)$$

$$\Longrightarrow i\mathcal{M}_3^{ab} = g^2\overline{v}(p_1)\gamma_\rho u(p_2)\frac{1}{k_3^2}f^{abc}T^c(k_3^\mu k_3^\rho - \eta^{\mu\rho}k_3^2)\varepsilon^*_\mu(k_1)$$
$$= -g^2\overline{v}(p_1)\gamma^\mu u(p_2)f^{abc}T^c\varepsilon^*_\mu(k_1),$$

where the first term vanishes because

$$\overline{v}(p_1)\gamma_\rho u(p_2)k_3^\rho = \overline{v}(p_1)\not{k}_3 u(p_2) = \overline{v}(p_1)(\underbrace{\not{k}_3+\not{p}_1}_{-\not{p}_2} - \not{p}_1 - m + m)u(p_2)$$
$$= \underbrace{\overline{v}(p_1)(-\not{p}_1 - m)}_{0}u(p_2) + \overline{v}(p_1)\underbrace{(-\not{p}_2+m)u(p_2)}_{0} = 0.$$

c) Aim: Compute $i\mathcal{M}_3^{ab}$ for $\varepsilon^*_\mu(k_1) = \varepsilon^+_\mu(k_1)$, $\varepsilon^*_\nu(k_2) = \varepsilon^-_\nu(k_2)$ with $\varepsilon^\pm(k) = \frac{1}{\sqrt{2}\,|\vec{k}|}\left(k^0, \pm\vec{k}\right)$.

In this setup, the polarization vector $\varepsilon^*_\nu(k_2)$ is still longitudinal, just of different length than in part b). We may thus quote our result from b) as it was just before using that $\varepsilon_\mu(k_1)$ is transverse:

$$i\mathcal{M}_3^{ab} = g^2\overline{v}(p_1)\gamma_\rho u(p_2)\frac{1}{k_3^2}f^{abc}T^c[-k_1^\mu k_1^\rho + k_3^\mu k_3^\rho - \eta^{\mu\rho}k_3^2]\frac{1}{\sqrt{2}\,|\vec{k}_2|}\varepsilon^+_\mu(k_1)$$
$$= \frac{1}{\sqrt{2}\,|\vec{k}_2|}i\mathcal{M}^{ab}_{3,\,\mathrm{b})}\big|_{\varepsilon^*_\mu(k_1)=\varepsilon^+_\mu(k_1)} - g^2\overline{v}(p_1)\gamma_\rho u(p_2)\frac{1}{k_3^2}f^{abc}T^c\underbrace{\frac{k_1^\mu k_1^\rho}{\sqrt{2}\,|\vec{k}_2|}\frac{k_{1\mu}}{\sqrt{2}\,|\vec{k}_1|}}_{k_1^\rho\,|\vec{k}_1|\,/\,|\vec{k}_2|,\text{ since }k_1^2=0},$$

where the $k_3^\mu k_3^\rho$-term again vanishes by the same argument and the $-\eta^{\mu\rho} k_3^2$-term gives just a scaled version of the result of part b).