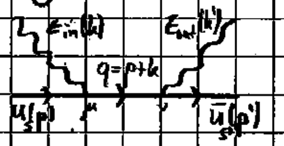
20.04.2015

Quantum Field Theory II - Assignment 1

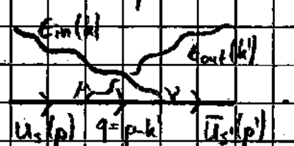
Problem 1.1 (Compton Scattering)

Consider the Compton scattering process $e^- + \gamma \rightarrow e^- + \gamma$ in QED. Use the Feynman rules to derive the amplitude for the tree level diagram.



$$= iM_1 = -ie^2 \bar{u}_s(p') \not\epsilon_{out}(k') \frac{\not p + \not k + m}{(p+k)^2 - m^2} \not\epsilon_{in}(k) u_s(p). \qquad (1)$$

Also compute the contribution from



The total amplitude, at order e^2, is the sum of these two diagrams. Show that if ϵ_{in} is replaced by the incoming photon momentum k, then the total amplitude vanishes. Check that the same holds true if ϵ_{out} is replaced by k'. Hint: You may find $(\not p - m) u_s(p) = 0$ useful.

$$iM_1 = \bar{u}_s(p)(-ie\gamma^\nu)\epsilon_\nu^{out}(k') \frac{i(\not q + m)}{q^2 - m^2} (-ie\gamma^\mu) \epsilon_\mu^{in}(k) u_s(p)$$

$$= -ie^2 \bar{u}_{s'}(p') \not\epsilon_{out}(k') \frac{\not p + \not k + m}{(p+k)^2 - m^2} \not\epsilon_{in}(k) u_s(p)$$

$$iM_2 = \bar{u}_s(p)(-ie\gamma^\nu)\epsilon_\nu^{in}(k) \frac{i(\not q + m)}{q^2 - m^2} (-ie\gamma^\mu) \epsilon_\mu^{out}(k') u_s(p)$$

$$= -ie^2 \bar{u}_{s'}(p') \not\epsilon_{in}(k) \frac{\not p - \not k' + m}{(p-k')^2 - m^2} \not\epsilon_{out}(k') u_s(p)$$

$$iM = i(M_1 + M_2) = -ie^2 \bar{u}_{s'}(p') \left(\not\epsilon_{out}(k') \frac{\not p + \not k + m}{(p+k)^2 - m^2} \not\epsilon_{in}(k) + \not\epsilon_{in}(k) \frac{\not p - \not k' + m}{(p-k')^2 - m^2} \not\epsilon_{out}(k') \right) u_s(p)$$

$$= -ie^2 \epsilon_{out,\mu}(k') \bar{u}_{s'}(p') \left(\gamma^\mu \frac{\not p + \not k + m}{(p+k)^2 - m^2} \gamma^\nu + \gamma^\nu \frac{\not p - \not k' + m}{(p-k')^2 - m^2} \gamma^\mu \right) \epsilon_{in,\nu}(k) u_s(p)$$

where in the last step, we used that $\epsilon_{in}(k)$, $\epsilon_{out}(k')$ are freely commutable through the remaining terms since they are spacetime 4-vectors and are therefore not contracted with any of the spinor space objects $u_s(p)$, $\bar{u}_{s'}(p')$ and the γ-matr.

Using the hint given in the exercise and applying momentum conservation, i.e. $p+k=p'+k'$, we get when setting $\epsilon_{in}(k)=k$

$$\not k\, u_s(p) \overset{\text{hint}}{=} (\not p+\not k-m)\,u_s(p)\;, \qquad \bar u_{s'}(p')\not k \overset{\text{m.c.}}{=} \bar u_{s'}(p')(\not p'+\not k'-\not p) = -\bar u_{s'}(p')(\not p-\not k'-m)$$

$$(\not p+\not k+m)(\not p+\not k-m) = (\not p+\not k)^2-m^2\;, \qquad (\not p-\not k'+m)(\not p-\not k'-m) = (\not p-\not k')^2-m^2$$

$$(\not p+\not k)^2 = \gamma^\alpha\gamma^\beta(p+k)_\alpha(p+k)_\beta = (2\eta^{\alpha\beta}-\gamma^\beta\gamma^\alpha)(p+k)_\alpha(p+k)_\beta = 2(p+k)_\alpha(p+k)^\alpha - \gamma^\beta(p+k)_\beta\gamma^\alpha(p+k)_\alpha$$

$$= 2(p+k)^2-(\not p+\not k)^2 \Longrightarrow (\not p+\not k)^2 = (p+k)^2, \text{ same for } (\not p-\not k')^2$$

Inserting everything into $i\mathcal{M}$

$$i\mathcal{M} = -ie^2\epsilon_{out,\mu}(k')\,\bar u_{s'}(p')\left(\gamma^\mu\frac{\not p+\not k+m}{(p+k)^2-m^2}\gamma^\nu+\gamma^\nu\frac{\not p-\not k'+m}{(p-k')^2-m^2}\gamma^\mu\right)k_\nu u_s(p)$$

$$= -ie^2\epsilon_{out,\mu}(k')\,\bar u_{s'}(p')\left(\gamma^\mu\frac{\not p+\not k+m}{(p+k)^2-m^2}(\not p+\not k-m)-(\not p-\not k'-m)\frac{\not p-\not k'+m}{(p-k')^2-m^2}\gamma^\mu\right)u_s(p)$$

$$= -ie^2\epsilon_{out,\mu}(k')\,\bar u_{s'}(p')\left(\gamma^\mu\frac{(p+k)^2-m^2}{(p+k)^2-m^2}-\frac{(p-k')^2-m^2}{(p-k')^2-m^2}\gamma^\mu\right)u_s(p)$$

$$= -ie^2\epsilon_{out,\mu}(k')\,\bar u_{s'}(p')\left(\gamma^\mu-\gamma^\mu\right)u_s(p) = 0$$

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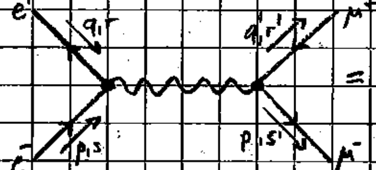
Problem 1.2 ($e^- + e^+ \to \mu^- + \mu^+$ - Scattering Amplitude)

For this question, use the fact that a muon, $\mu^\pm$, is a Dirac Fermion with mass $m_\mu \gg m_e$ and satisfies the same Feynman rules as the electron.

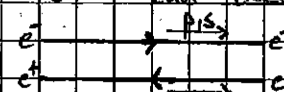
a) Using the Feynman rules for QED, show that the amplitude for $e^- + e^+ \to \mu^- + \mu^+$ is given, at lowest order in e, by



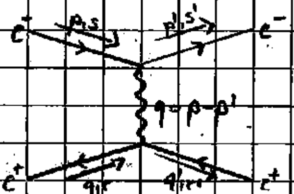
$$= i\mathcal{M} = +ie^2\, \bar{v}^e_r(q)\, \gamma_\mu\, u^e_s(p)\, \frac{1}{(p+q)^2 + i\varepsilon}\, \bar{u}^\mu_{s'}(p')\, \gamma^\mu\, v^\mu_{r'}(q') \quad (2)$$

where the superscripts e and m denote whether the spinors satisfy the Dirac equation for electrons or muons. Briefly comment on why this is the only contributing diagram unlike for $e^- + e^+ \to e^- + e^+$-scattering.

Overlooking the fact that the event $e^- + e^+ \to e^- + e^+$ can occur simply due to trivial scattering (i.e. the diagram



) because initial and final particle composition are identical, we note that for this very same reason, $e^- + e^+ \to e^- + e^+$ receives contributions from two non-trivial scattering channel. The diagram in eq. (2) works for $e^- + e^+ \to e^- + e^+$ as well as



Such a t-channel diagram cannot contribute to $e^- + e^+ \to \mu^- + \mu^+$ as it is incapable of altering the particle composition.

b) Prove the following identities:

i) $\mathrm{Tr}(\gamma^\mu \gamma^\nu) = 4\eta^{\mu\nu}$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = \mathrm{Tr}(\{\gamma^\mu, \gamma^\nu\} - \gamma^\nu\gamma^\mu) = \mathrm{Tr}(2\eta^{\mu\nu}\mathbb{1}_4) - \mathrm{Tr}(\gamma^\nu\gamma^\mu)$$

$$= 2\eta^{\mu\nu}\,\mathrm{Tr}(\mathbb{1}_4) - \mathrm{Tr}(\gamma^\mu\gamma^\nu) = 8\eta^{\mu\nu} - \mathrm{Tr}(\gamma^\mu\gamma^\nu)$$

$$\Longrightarrow \quad \mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu}$$

ii) $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\sigma) = 0$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\sigma) = \mathrm{Tr}(\underbrace{(\gamma^5)^2}_{\mathbb{1}_4}\gamma^\mu\gamma^\nu\gamma^\sigma) = -\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^5\gamma^\nu\gamma^\sigma) = (-1)^3\,\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^5) = -\mathrm{Tr}((\gamma^5)^2\gamma^\mu\gamma^\nu\gamma^\sigma)$$

$$\Longrightarrow \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\sigma) = 0$$

iii) $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho) = 4(\eta^{\mu\nu}\eta^{\sigma\rho} - \eta^{\mu\sigma}\eta^{\nu\rho} + \eta^{\mu\rho}\eta^{\nu\sigma})$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho) = -\mathrm{Tr}(\gamma^\nu\gamma^\mu\gamma^\sigma\gamma^\rho) + 2\eta^{\mu\nu}\mathrm{Tr}(\gamma^\sigma\gamma^\rho) = \mathrm{Tr}(\gamma^\nu\gamma^\sigma\gamma^\mu\gamma^\rho) + 8\eta^{\mu\nu}\eta^{\sigma\rho}$$

$$- 2\eta^{\mu\sigma}\mathrm{Tr}(\gamma^\nu\gamma^\rho) = -\mathrm{Tr}(\gamma^\nu\gamma^\sigma\gamma^\rho\gamma^\mu) + 8\eta^{\mu\nu}\eta^{\sigma\rho} - 8\eta^{\mu\sigma}\eta^{\nu\rho}$$

$$+ 2\eta^{\mu\rho}\mathrm{Tr}(\gamma^\nu\gamma^\sigma) = -\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho) + 8(\eta^{\mu\nu}\eta^{\sigma\rho} - \eta^{\mu\sigma}\eta^{\nu\rho} + \eta^{\mu\rho}\eta^{\nu\sigma})$$

$$\Longrightarrow \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho) = 4(\eta^{\mu\nu}\eta^{\sigma\rho} - \eta^{\mu\sigma}\eta^{\nu\rho} + \eta^{\mu\rho}\eta^{\nu\sigma})$$

iv) $\sum_{s,s'} [\bar{v}_{s'}(p')\gamma^\nu u_s(p)]^* [\bar{v}_{s'}(p')\gamma^\mu u_s(p)] = 4[p^\nu p'^\mu + p^\mu p'^\nu - (pp' + m^2)\eta^{\mu\nu}]$

Hint: You may find the following relations useful,

$$\sum_s u_s(p)\bar{u}_s(p) = \gamma p + m, \qquad \sum_s v_s(p)\bar{v}_s(p) = \gamma p - m.$$

$$\sum_{s,s'} [\bar{v}_{s'}(p')\gamma^\nu u_s(p)]^* [\bar{v}_{s'}(p')\gamma^\mu u_s(p)] = \sum_{s,s'} [\underbrace{u_s^\dagger(p)\gamma^0}_{\bar{u}_s(p)}\underbrace{\gamma^0\gamma^{\nu\dagger}\gamma^0}_{\gamma^\nu} v_{s'}(p')][\bar{v}_{s'}(p')\gamma^\mu u_s(p)]$$

$$= \sum_s \bar{u}_s(p)\gamma^\nu \underbrace{\sum_{s'} v_{s'}(p')\bar{v}_{s'}(p')}_{\gamma p' - m}\gamma^\mu u_s(p) = \sum_s (\bar{u}_s(p))_\alpha (\gamma^\nu)_{\alpha\beta} (\gamma p' - m)_{\beta\gamma} (\gamma^\mu)_{\gamma\delta} (u_s(p))_\delta$$

$$\overset{*}{=} \mathrm{Tr}(\gamma^\nu(\gamma^\rho p'_\rho - m)\gamma^\mu(\gamma^\sigma p_\sigma + m)) = p'_\rho p_\sigma \mathrm{Tr}(\gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma) - m^2\mathrm{Tr}(\gamma^\nu\gamma^\mu)$$

$$= p'_\rho p_\sigma 4(\eta^{\nu\rho}\eta^{\mu\sigma} - \eta^{\nu\mu}\eta^{\rho\sigma} + \eta^{\nu\sigma}\eta^{\rho\mu}) - m^2 4\eta^{\nu\mu}$$

$$= 4(p^\mu p'^\nu - \eta^{\nu\mu}p'_\rho p^\rho + p^\nu p'^\mu - m^2\eta^{\nu\mu}) = 4[p^\mu p'^\nu + p^\nu p'^\mu - (pp' + m^2)\eta^{\mu\nu}]$$

c) Let m, M denote the electron and muon masses, respectively. Show that

$$\sum_{s,r,s',r'} |\mathcal{M}|^2 = \frac{e^4}{s^2}\mathrm{Tr}[(\gamma p' + M)\gamma^\mu(\gamma q' - M)\gamma^\nu]\,\mathrm{Tr}[(\gamma p + m)\gamma_\nu(\gamma q - m)\gamma_\mu], \quad (4)$$

where $s = (p+q)^2$.

* all spinor indices summed over, hence result is a number, therefore, adding a trace changes nothing

$$|\mathcal{M}|^2 = \mathcal{M}^* \mathcal{M} = -ic^2 \left[\bar v_r^e(q)\gamma_\mu u_s^e(p)\right]^* \frac{1}{(p+q)^2} \left[\bar u_{s'}^m(p')\gamma^\mu v_{r'}^m(q')\right]^*$$

$$\cdot\, ic^2 \left[\bar v_r^e(q)\gamma_\nu u_s^e(p)\right] \frac{1}{(p+q)^2}\left[\bar u_{s'}^m(p')\gamma^\nu v_{r'}^m(q')\right]$$

$$= \frac{e^4}{s^2}\left[u_s^{\dagger e}(p) \underbrace{\gamma^0\gamma^0}_{1_4} \gamma_\mu^\dagger \gamma^0 v_r^{e\dagger}(q)\right]\left[v_{r'}^{m\dagger}(q')\underbrace{\gamma^0\gamma^0}_{1_4}\gamma^{\mu\dagger}\gamma^0 u_{s'}^{m\dagger}(p')\right]$$

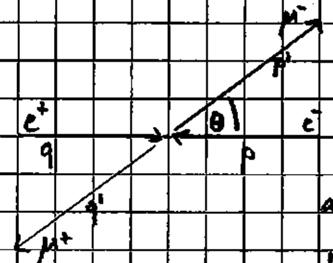
$$\left[\bar v_r^e(q)\gamma_\nu u_s^e(p)\right]\left[\bar u_{s'}^m(p')\gamma^\nu v_{r'}^m(q')\right]$$

$$= \frac{e^4}{s^2}\left[\bar u_s^e(p)\gamma_\mu v_r^e(q)\right]\left[\bar v_{r'}^m(q')\gamma^\mu u_{s'}^m(p')\right]\left[\bar v_r^e(q)\gamma_\nu u_s^e(p)\right]\left[\bar u_{s'}^m(p')\gamma^\nu v_{r'}^m(q')\right]$$

$$\sum_{s,r,s',r'} |\mathcal{M}|^2 = \sum_{s,r,s',r'} \frac{e^4}{s^2}\left[\bar v_r^e(q)\gamma_\mu u_s^e(p)\right]^*\left[\bar u_{s'}^m(p')\gamma^\mu v_{r'}^m(q')\right]^*\left[\bar v_r^e(q)\gamma_\nu u_s^e(p)\right]\left[\bar u_{s'}^m(p')\gamma^\nu v_{r'}^m(q')\right]$$

$$\overset{b)(ii)}{=} \frac{e^4}{s^2}\, Tr\left[(\gamma p' + M)\gamma^\mu(\gamma q' - M)\gamma^\nu\right]\, Tr\left[(\gamma p + m)\gamma_\nu(\gamma q - m)\gamma_\mu\right]$$

d) This can be simplified further, assuming that the momentum components are sufficiently large enough and thus, one can neglect the electron and muon masses as a good approximation.



In the centre-of-mass frame,

$$\vec q = -\vec p, \qquad \vec q\,' = -\vec p\,' \tag{5}$$

$$\text{and} \quad q^0 = p^0 = |\vec p|, \qquad q'^0 = p'^0 = |\vec p\,'| \tag{6}$$

by setting $m = M = 0$. Show that

$$\sum_{s,r,s',r'} |\mathcal{M}|^2 = \frac{32e^4}{s^2}\left[p\cdot p'\, q\cdot q' + p\cdot q'\, q\cdot p'\right] = 4e^4(1+\cos^2\theta), \tag{7}$$

where θ is the scattering angle in the centre-of-mass frame.

By setting $m = M = 0$, eq. (4) becomes

$$\sum_{s,r,s',r'} |\mathcal{M}|^2 = \frac{e^4}{s^2} p'_\sigma q'_\rho\; Tr\left[\gamma^\sigma\gamma^\mu\gamma^\rho\gamma^\nu\right] p^\sigma q^\rho\; Tr\left[\gamma_\sigma\gamma_\nu\gamma_\rho\gamma_\mu\right]$$

$$\overset{b)(iii)}{=} \frac{e^4}{s^2} p'_\sigma q'_\rho\, 4\left(\eta^{\sigma\mu}\eta^{\rho\nu} - \eta^{\sigma\rho}\eta^{\mu\nu} + \eta^{\sigma\nu}\eta^{\mu\rho}\right) p^\sigma q^\rho\, 4\left(\eta_{\sigma\nu}\eta_{\rho\mu} - \eta_{\sigma\rho}\eta_{\nu\mu} + \eta_{\sigma\mu}\eta_{\nu\rho}\right)$$

$$= \frac{16e^4}{s^2}\left(p'^\mu q'^\nu - p'^\rho q'_\rho \eta^{\mu\nu} + p'^\nu q'^\mu\right)\left(p_\nu q_\mu - p_\rho q^\rho \eta_{\nu\mu} + p_\mu q_\nu\right)$$

$$= \frac{16e^4}{s^2}\left(p'\cdot q\, q'\cdot p + p\cdot q\, p'\cdot q' + p'\cdot p\, q'\cdot q - p'\cdot q'\, p\cdot q + 4\, p\cdot q'\, p\cdot q - p'\cdot q'\, p\cdot q + p'\cdot p\, q'\cdot q\right.$$
$$\left. - p'\cdot q'\, p\cdot q + p'\cdot q\, q'\cdot p\right) = \frac{32e^4}{s^2}\left(p\cdot p'\, q\cdot q' + p\cdot q'\, p'\cdot q\right)$$

Now, in the centre-of-mass frame, we can concretize

$$s^2 = (p+q)^4 = (p+q)^2(p'+q')^2 = \Big[\underbrace{(p^0+q^0)^2}_{2|\vec{p}|} - \underbrace{(\vec{p}+\vec{q})^2}_{0}\Big]\Big[\underbrace{(p'^0+q'^0)^2}_{2|\vec{p}'|} - \underbrace{(\vec{p}'+\vec{q}')^2}_{0}\Big] = 16|\vec{p}|^2|\vec{p}'|^2$$

$$p\cdot p'\, q\cdot q' = (p^0 p'^0 - \vec{p}\cdot\vec{p}')(q^0 q'^0 - \vec{q}\cdot\vec{q}') = (|\vec{p}||\vec{p}'| - |\vec{p}||\vec{p}'|\cos\theta)(|\vec{p}||\vec{p}'| - |\vec{p}||\vec{p}'|\cos\theta)$$

$$= |\vec{p}|^2|\vec{p}'|^2(1 - 2\cos\theta + \cos^2\theta)$$

$$p\cdot q'\, q\cdot p' = (p^0 q'^0 + \vec{p}\vec{p}')(q^0 p'^0 + \vec{q}\vec{q}') = (|\vec{p}||\vec{p}'| + |\vec{p}||\vec{p}'|\cos\theta)(|\vec{p}||\vec{p}'| + |\vec{p}||\vec{p}'|\cos\theta)$$

$$= |\vec{p}|^2|\vec{p}'|^2(1 + 2\cos\theta + \cos^2\theta)$$

Assembling the above, we get

$$\sum_{s,r,s',r'} |M|^2 = \frac{32e^4}{s^2}\left[p\cdot p'\, q\cdot q' + p\cdot q'\, q\cdot p'\right] = \frac{32e^4}{16|\vec{p}|^2|\vec{p}'|^2}\Big[|\vec{p}|^2|\vec{p}'|^2(1 - 2\cos\theta + \cos^2\theta)$$
$$+ |\vec{p}|^2|\vec{p}'|^2(1 + 2\cos\theta + \cos^2\theta)\Big]$$

$$= \frac{2e^4}{|\vec{p}|^2|\vec{p}'|^2}\, 2|\vec{p}|^2|\vec{p}'|^2(1+\cos^2\theta) = 4e^4(1+\cos^2\theta)$$

Problem 1.3 (Symmetries of classical electrodynamics)

We consider classical massless electrodynamics

$$\mathcal{L}_{ED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi} \gamma^\mu (i\partial_\mu + A_\mu) \Psi, \tag{8}$$

where the field strength is defined as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 2\partial_{[\mu} A_{\nu]}$.
This theory is invariant under local $U(1)$ gauge transformations, with the fields transforming as,

$$\begin{aligned} \Psi(x) &\longrightarrow e^{i\alpha(x)} \Psi(x), \\ A_\mu(x) &\longrightarrow A_\mu(x) + \partial_\mu \alpha(x). \end{aligned} \tag{9}$$

a) Find the equations of motion for the gauge and fermion fields.

We derive the equations of motion by applying the principle of stationary action.

$\bar{\Psi}$: $$0 \stackrel{!}{=} \frac{\delta S}{\delta \bar{\Psi}(y)} = \frac{\delta}{\delta \bar{\Psi}(y)} \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi}(x) (i\partial_\mu + A_\mu) \Psi(x) \right)$$

$$= \int d^4x \underbrace{\frac{\delta \bar{\Psi}(x)}{\delta \bar{\Psi}(y)}}_{\delta^{(4)}(x-y)} \gamma^\mu (i\partial_\mu + A_\mu) \Psi(x) = \gamma^\mu (i\partial_\mu + A_\mu) \Psi(x)$$

Ψ: the e.o.m. for $\Psi(y)$ is obtained most easily by complex conjugation of the above

$$\left(\gamma^\mu (i\partial_\mu + A_\mu) \Psi(x) \right)^* = \Psi^\dagger(x) (-i\partial_\mu + A_\mu) \gamma^{\mu\dagger} = \Psi^\dagger(x) (-i\partial_\mu + A_\mu) \gamma^0 \gamma^\mu \gamma^0 = 0$$

multiplication with γ^0 from the right gives: $\bar{\Psi}(x)(-i\partial_\mu + A_\mu)\gamma^\mu = 0$

A_μ: $$0 \stackrel{!}{=} \frac{\delta S}{\delta A_\rho(y)} = \frac{\delta}{\delta A_\rho(y)} \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi} \gamma^\mu (i\partial_\mu + A_\mu(x)) \Psi \right)$$

here one needs to be diligent, when varying the first term

$$\frac{\delta}{\delta A_\rho(y)} \left(-\frac{1}{4} F^2 \right) = -\frac{1}{2} F^{\mu\nu} \frac{\delta}{\delta A_\rho(y)} \left(2\partial_{[\mu} A_{\nu]} \right) = -F^{\mu\nu} \partial_{[\mu} \delta^\rho_{\nu]} \delta^{(4)}(x-y)$$

$$0 \stackrel{!}{=} \frac{\delta S}{\delta A_\rho(y)} = \int d^4x \left(-F^{\mu\nu} \partial_{[\mu} \delta^\rho_{\nu]} \delta^{(4)}(x-y) + \bar{\Psi} \gamma^\mu \underbrace{\frac{\delta A_\mu(x)}{\delta A_\rho(y)}}_{\delta_\mu^\rho \delta^{(4)}(x-y)} \Psi \right.$$

$$= \int d^4x \left(-\frac{1}{2} (\partial_\mu F^{\mu\rho} - \partial_\nu F^{\rho\nu}) + \bar{\Psi} \gamma^\rho \Psi \right) \delta^{(4)}(x-y) = -\partial_\mu F^{\mu\rho} + \bar{\Psi} \gamma^\rho \Psi = 0$$

b) The energy-momentum tensor, T^μ_ν, is the conserved Noether current associated with the space-time translations

$$x^\mu \longrightarrow x^{\mu\prime} = x^\mu + a^\nu \delta_\nu{}^\mu, \tag{10}$$

with the fields transforming as

$$\begin{aligned} A_\mu(x) &\longrightarrow A'_\mu(x') = A_\mu(x), \\ \psi(x) &\longrightarrow \psi'(x') = \psi(x). \end{aligned} \tag{11}$$

i) Show that, for classical electrodynamics, the energy-momentum tensor is given by

$$T^\mu{}_\nu = -F^{\mu\rho}\partial_\nu A_\rho + \delta^\mu{}_\nu\left(\tfrac{1}{4}F_{\rho\sigma}F^{\rho\sigma}\right) + i\bar{\psi}\gamma^\mu\partial_\nu\psi. \tag{12}$$

Hint: Noether's theorem states that given a symmetry transformation parametrised by ε^a (a runs over the number of independent transformations, e.g. $a \in \{1\}$ for a $U(1)$ symmetry or $a = \nu \in \{0,1,2,3\}$ for spacetime translations) inducing the infinitesimal transformations

$$\begin{aligned} x^\mu &\longrightarrow x^{\mu\prime} = x^\mu - \varepsilon^a \mathcal{E}^\mu{}_a + \mathcal{O}(\varepsilon^2), \\ \chi_i(x) &\longrightarrow \chi_i'(x) = \chi_i(x) + \varepsilon^a \Delta_{ai} + \mathcal{O}(\varepsilon^2), \quad [\text{note: } \chi'(x), \text{ not } \chi'(x')] \end{aligned} \tag{13}$$

where χ_i is any field depending on the transformed quantity, there exists a conserved current,

$$J^\mu{}_a = \frac{\partial\mathcal{L}}{\partial(\partial_\mu \chi_i)}\Delta_{ai} - \mathcal{E}^\mu{}_a \mathcal{L}. \tag{14}$$

First, we remark that Noether's theorem is only valid on-shell, i.e. after imposing the equations of motion. Now, comparing our transform. to the one given in the hint, we see that the latter can be converted into the former via the assignments

$$x^\mu \to x'^\mu = x^\mu - a^\nu \delta^\mu{}_\nu, \quad x^\mu \to x'^\mu = x^\mu - \varepsilon^a \mathcal{E}^\mu{}_a + \mathcal{O}(\varepsilon^2) \implies \begin{cases} \varepsilon^a \longrightarrow a^\nu \\ \mathcal{E}^\mu{}_a \longrightarrow \delta^\mu{}_\nu \end{cases}$$

$$A_\mu(x) \longrightarrow A'_\mu(x) = A_\mu(x) \implies A'_\mu(x) = A_\mu(T^{-1}x) = A_\mu(x+a) \left(\overset{!}{=} A_\mu(x) + \partial^\nu \Delta_\nu\right),$$

$$A'_\mu(x) = A_\mu(x) + \partial^\nu \underbrace{\partial_\nu A_\mu(x)}_{(\Delta^{(A)})^\mu{}_\nu} + \mathcal{O}(a^2),$$

where T marks our spacetime transformation and T^{-1} its inverse/back-transformation. Doing the same for $\Psi(x)$, we see

$$\Psi(x) \longrightarrow \Psi'(x') = \Psi(x) \implies \Psi'(x) = \Psi(T^{-1}x) = \Psi(x+a)$$
$$= \Psi(x) + \partial^\nu \underbrace{\partial_\nu \Psi(x)}_{\Delta^\Psi_\nu} + \mathcal{O}(a^2)$$

Therefore, $(\Delta^A)^\mu{}_\nu = \partial_\nu A^\mu$ and $\Delta^\Psi_\nu = \partial_\nu \Psi$.

With this, we may assemble the energy-momentum tensor

$$T^\mu{}_\nu = \sum_{\substack{\text{sum over}\\ \text{field d.o.f.}}} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \chi_i)} \Delta_{a i} - \delta^\mu{}_a \mathcal{L}$$

$$\longrightarrow \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)} (\Delta^A)_{\rho\nu} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi)} \Delta^\Psi_\nu + \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\Psi})}}_{0} \Delta^{\bar{\Psi}}_\nu - \delta^\mu{}_\nu \mathcal{L},$$

where $$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)} = -\frac{1}{2} F^{\alpha\beta} \frac{\partial}{\partial(\partial_\mu A_\rho)} (\partial_\alpha A_\beta - \partial_\beta A_\alpha) = -\frac{1}{2} F^{\alpha\beta} (\delta^\mu_\alpha \delta^\rho_\beta - \delta^\mu_\beta \delta^\rho_\alpha)$$
$$= -\frac{1}{2}(F^{\mu\rho} - F^{\rho\mu}) = -F^{\mu\rho}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi)} = i\bar{\Psi}\gamma^\mu$$

$$T^\mu{}_\nu = -F^{\mu\rho}\partial_\nu A_\rho + i\bar{\Psi}\gamma^\mu \partial_\nu \Psi + \frac{1}{4}\delta^\mu{}_\nu F_{\alpha\beta}F^{\alpha\beta} - \delta^\mu{}_\nu \underbrace{\bar{\Psi}\gamma^\rho(i\partial_\rho + A_\rho)\Psi}_{0,\ \text{using e.o.m.}}$$

$$= \underline{\underline{-F^{\mu\rho}\partial_\nu A_\rho + \frac{1}{4}\delta^\mu{}_\nu F_{\alpha\beta}F^{\alpha\beta} + i\bar{\Psi}\gamma^\mu\partial_\nu\Psi}}$$

ii) Show that $T^\mu{}_\nu$ is not gauge invariant.

Under the local $U(1)$ gauge transformation, we have

$$\Psi(x) \longrightarrow e^{i\alpha(x)}\Psi(x), \qquad A_\mu(x) \longrightarrow A_\mu + \partial_\mu \alpha(x).$$

We see immediately that transforming Ψ does not change $T^\mu{}_\nu$. Under the $A_\mu(x)$-transformation, however, we obtain

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \longrightarrow \partial_\mu A_\nu + \partial_\mu\partial_\nu \alpha(x) - \partial_\nu A_\mu - \partial_\nu \partial_\mu \alpha(x)$$
$$= F_{\mu\nu} + \underbrace{(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)}_{0} \alpha(x),$$

where the derivatives vanish by Schwarz' theorem since $\alpha(x)$ is taken to have continuous second partial derivatives at all points in spacetime. So apparently, only the first of $T^\mu{}_\nu$ changes under a local $U(1)$ gauge transformation,

$$T^\mu{}_\nu \to T^\mu{}_\nu + \partial_\nu \partial_\rho \alpha(x) \neq T^\mu{}_\nu$$

which can therefor not be offset, rendering $T^\mu{}_\nu$ gauge variant.

c) We may restore gauge invariance by 'improving' the energy-momentum tensor. A conserved current, $J^\mu{}_a$, can always be improved with the help of a (non-conserved) antisymmetric tensor, $L^{\mu\nu}{}_a = -L^{\nu\mu}{}_a$.

i) Show that the improved current

$$\tilde{J}^\mu{}_a = J^\mu{}_a + \partial_\nu L^{\mu\nu}{}_a \qquad (15)$$

is also conserved and gives rise to the same conserved charge, $\hat{Q}_a$, as that of $J^\mu{}_a$. Hint: Recall that $Q_a = \int d^3x\, J^0{}_a$.

$\tilde{J}^\mu{}_a$ is conserved if its total derivative, $\partial_\mu \tilde{J}^\mu{}_a$, vanishes. This is the case.

$$\partial_\mu \tilde{J}^\mu{}_a = \partial_\mu (J^\mu{}_a + \partial_\nu L^{\mu\nu}{}_a) = \underbrace{\partial_\mu J^\mu{}_a}_{0} + \underbrace{\partial_\mu \partial_\nu L^{\mu\nu}{}_a}_{0} = 0,$$

where the first term vanishes because $J^\mu{}_a$ is conserved and the second because it is a fully contracted sum antisymmetric under the exchange of indices, i.e.

$$\partial_\mu \partial_\nu L^{\mu\nu}{}_a = -\partial_\nu \partial_\mu L^{\nu\mu}{}_a \overset{\mu \leftrightarrow \nu}{=} -\partial_\mu \partial_\nu L^{\mu\nu}{}_a \implies \partial_\mu \partial_\nu L^{\mu\nu}{}_a = 0$$

$$\hat{Q}_a = \int d^3x\, \tilde{J}^0_a = \underbrace{\int d^3x\, J^0_a}_{Q_a} + \int d^3x\, \partial_\nu L^{0\nu}{}_a = Q_a + \underbrace{\int d^3x\, \partial_0 L^{00}{}_a}_{0} - \underbrace{\int d^3x\, \partial_i L^{0i}{}_a}_{0}$$

Here, $L^{00} = 0$ because $L^{\mu\nu}{}_a = -L^{\nu\mu}{}_a$. The last term above vanishes, after applying Gauß' theorem to turn it into a surface integral at infinity, where we take our current to be zero.

ii) Now, focussing on the limit $\psi, \bar{\psi} \to 0$ for simplicity, use the equations of motion to find the antisymmetric tensor $L^{\mu\ \rho}_{\ \nu}$ that improves the energy-momentum tensor, restoring gauge invariance, i.e.

$$\Theta^\mu_{\ \nu}\Big|_{\psi=\bar{\psi}=0} = \left(T^\mu_{\ \nu} + \partial_\rho L^{\mu\ \rho}_{\ \nu}\right)\Big|_{\psi=\bar{\psi}=0} = -F^{\mu\rho}F_{\nu\rho} + \frac{1}{4}\delta^\mu_{\ \nu} F_{\rho\sigma}F^{\rho\sigma} \qquad (16)$$

In the limit $\psi, \bar{\psi} \to 0$, our energy-momentum tensor becomes

$$T^\mu_{\ \nu} = -F^{\mu\rho}\partial_\nu A_\rho + \frac{1}{4}\delta^\mu_{\ \nu} F_{\rho\sigma}F^{\rho\sigma} + i\bar{\psi}\gamma^\mu\partial_\mu\psi \xrightarrow{\psi\bar{\psi}\to 0} -F^{\mu\rho}\partial_\nu A_\rho + \frac{1}{4}\delta^\mu_{\ \nu}F_{\rho\sigma}F^{\rho\sigma}$$

while the equation of motion for $A_\mu(x)$ simplifies to

$$-\partial_\mu F^{\mu\nu} + \bar{\psi}\gamma^\nu\psi \longrightarrow \partial_\mu F^{\mu\nu} = 0$$

To find $L^{\mu\ \rho}_{\ \nu}$, we insert our old energy-momentum tensor $T^\mu_{\ \nu}$ in the $\psi, \bar{\psi} \to 0$ - limit into the improved energy-momentum tensor and solve for $L^{\mu\ \rho}_{\ \nu}$.

$$\Theta^\mu_{\ \nu} = T^\mu_{\ \nu} + \partial_\rho L^{\mu\ \rho}_{\ \nu} = -F^{\mu\rho}\partial_\nu A_\rho + \frac{1}{4}\delta^\mu_{\ \nu}F_{\rho\sigma}F^{\rho\sigma} + \partial_\rho L^{\mu\ \rho}_{\ \nu}$$

$$\stackrel{!}{=} -F^{\mu\rho}F_{\nu\rho} + \frac{1}{4}\delta^\mu_{\ \nu}F_{\rho\sigma}F^{\rho\sigma} \qquad \text{(s. eq. (16))}$$

$$\Longrightarrow \partial_\rho L^{\mu\ \rho}_{\ \nu} = -F^{\mu\rho}F_{\nu\rho} + F^{\mu\rho}\partial_\nu A_\rho = F^{\mu\rho}\partial_\rho A_\nu$$

$$= F^{\mu\rho}\partial_\rho A_\nu + \underbrace{\partial_\rho F^{\rho\mu}}_{0,\ \text{e.o.m.}} A_\nu = \partial_\rho \big(\underbrace{F^{\mu\rho}A_\nu}_{L^{\mu\ \rho}_{\ \nu}}\big)$$

d) Finally, massless electrodynamics has one additional symmetry: spacetime symmetries are enhanced with scale invariance,

$$x^\mu \longrightarrow x'^\mu = e^{-\beta}x^\mu,$$
$$A_\mu(x) \longrightarrow A'_\mu(x') = e^{\beta}A_\mu(x), \qquad \beta \in \mathbb{R} \qquad (17)$$
$$\psi(x) \longrightarrow \psi'(x') = e^{\frac{3}{2}\beta}\psi(x) \qquad \left[\text{more general } \chi_i(x) \longrightarrow \chi'_i(x') = e^{\dim[\chi_i]\beta}\chi_i(x)\right]$$

in addition to the usual Poincaré invariance.

i) Show that the action $S=\int d^4x\, \mathcal{L}_{ED}$ is invariant under scale transf.

The electrodynamics Lagrangian $\mathcal{L}_{ED}$ as given in eq. (8),

$$\mathcal{L}_{ED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi} \gamma^\mu (i\partial_\mu + A_\mu) \Psi,$$

can be seen to transform like $\mathcal{L}_{ED} \to \mathcal{L}'_{ED} = e^{4\beta} \mathcal{L}_{ED}$ under scaling most easily by looking at its last term,

$$\bar{\Psi} \gamma^\mu A_\mu \Psi \longrightarrow \bar{\Psi}' \gamma^\mu A'_\mu \Psi' = e^{\frac{3}{2}\beta} \bar{\Psi} \gamma^\mu e^\beta A_\mu e^{\frac{3}{2}\beta} \Psi = e^{4\beta} \bar{\Psi} \gamma^\mu A_\mu \Psi.$$

Therefore the action S as a whole is unaffected by scaling,

$$S \longrightarrow S' = \int d^4x' \mathcal{L}'_{ED} = \int d^4x\, e^{-4\beta} e^{4\beta} \mathcal{L}_{ED} = \int d^4x\, \mathcal{L}_{ED} = S$$

ii) Show that the associated current, S^μ, can be written as

$$S^\mu = x^\nu T^\mu{}_\nu + U^\mu, \qquad (18)$$

and give an expression for U^μ.

S^μ is constructed by calculating the following terms

$$S^\mu = \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}}_{-F^{\mu\nu}} \Delta^{A_\mu}_\nu + \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi)}}_{i\bar{\Psi}\gamma^\mu} \Delta^\Psi + \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\Psi})}}_{0} \Delta^{\bar{\Psi}} - x^\mu \mathcal{L}$$

To obtain $\Delta^{A_\mu}_\nu$ and Δ^Ψ, we need to know how the fields transform. By Taylor expanding the transformation rules in eq. (17) to linear order, we get

$$A'_\mu(x) = e^\beta A_\mu(T^{-1}x) = e^\beta A_\mu(e^\beta x) = A_\mu(x) + \beta\left(A_\mu(x) + x^\nu \partial_\nu A_\mu(x)\right) + \mathcal{O}(\beta^2)$$

$$\Psi'(x) = e^{\frac{3}{2}\beta} \Psi(T^{-1}x) = e^{\frac{3}{2}\beta} \Psi(e^\beta x) = \Psi(x) + \beta\left(\frac{3}{2}\Psi(x) + x^\nu \partial_\nu \Psi(x)\right) + \mathcal{O}(\beta^2)$$

Therefore, $\Delta^{A_\mu}_\mu(x^\mu) = A_\mu(x) + x^\nu \partial_\nu A_\mu(x) = A_\mu(x) + x^\nu \Delta^{EMT}_{\mu\nu}$ and
$\Delta^\Psi = \frac{3}{2}\Psi(x) + x^\nu \partial_\nu \Psi(x) = \frac{3}{2}\Psi(x) + x^\nu \Delta^{EMT}_\nu$ (EMT: electromagnetic tensor $F^{\mu\nu}$)

$$S^\mu = -F^{\mu\nu}(A_\nu + x^\rho \Delta^{EMT}_{\nu\rho}) + i\bar{\Psi}\gamma^\mu\left(\frac{3}{2}\Psi + x^\nu \Delta^{EMT}_\nu\right) - x^\mu \mathcal{L}$$
$$= -F^{\mu\nu} A_\nu + \frac{3}{2} \bar{\Psi}\gamma^\mu \Psi + x^\nu\left(-F^{\mu\nu}\Delta^{EMT}_{\mu\nu} + i\bar{\Psi}\gamma^\mu \Delta^{EMT}_\nu - \delta^\mu{}_\nu \mathcal{L}\right) = U^\mu + x^\nu T^\mu{}_\nu$$