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General Relativity - Exercise Sheet 1

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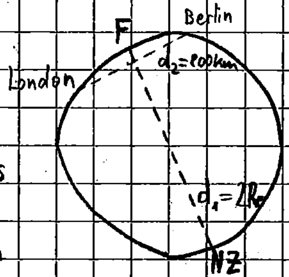
1. Gravitational Train (10 points)

In order to save CO_2-emissions from air traffic, the governments of France and New Zealand have decided, very realistically, to bore a tunnel through the center of the Earth, directly connecting their land masses with a straight tube, through which a train can travel without any friction and solely accelerated by grav. pull.



a) How long does it take for the train to travel one way if Earth's radial density $\rho(r)$ follows

i) $\rho(r) = \rho_0 = 5{,}5\,\frac{g}{cm^3}$, where $0 \leq r \leq R_E$?

The gravitational pull on the train depends on its current distance r from the Earth's core, since only the Earth's mass below it, i.e. $M(r) = \frac{4}{3}\pi\rho_0 r^3$, contributes. The force is therefore given by

$$F_g = -G\,\frac{M(r)\,m_{train}}{r^2} = -\frac{4}{3}\pi G\rho_0 \cdot m_{train}\, r\,,$$

which yields the equation of motion

$$m_{train}\,\ddot{r} = F_g = -\underbrace{\frac{4}{3}\pi G\rho_0}_{k^2}\, m_{train}\, r = -m_{train}\, k^2 r \implies \ddot{r} = -k^2 r$$

Unsurprisingly, we find that a train in a tube through Earth's centre without friction behaves like a harmonic oscillator, bouncing back and forth between France and New Zealand.

The period P, in this case twice the travel time t, of a harmonic oscillator depends solely on its suspension rate k such that

$$P = \frac{2\pi}{k} = 2t\,, \qquad t = \frac{\pi}{k} = \frac{\pi}{\sqrt{\frac{4}{3}\pi G\rho_0}} = \sqrt{\frac{3\pi}{4G\rho_0}} \approx 2530\,s \approx 42{,}2\,min$$

Interestingly, the travel time depends only on the Earth's density ρ_0

ii) $\rho(r) = \rho_1 \frac{R_E}{r}$ where $\rho_1 = 3{,}7 \frac{g}{cm^3}$ and $0 < r \leq R_E$.

In this case, $M(r) = \frac{4}{3}\pi r^3 \rho(r) = \frac{4}{3}\pi R_E \rho_1 r^2$ and

$$F_g = -G\frac{M(r)\, m_{train}}{r^2} = -\frac{4}{3}\pi R_E \rho_1 G\, m_{train}$$ f Integral needed.

is independent of r. Under a constant force, the acceleration

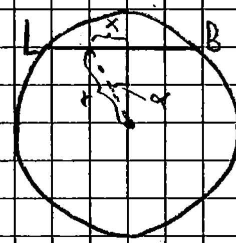
$$a = \frac{4}{3}\pi R_E \rho_1 G$$

is also constant, rendering the distance travelled easily calculatable via $s = \frac{a}{2}t^2$. Solving for t and inserting $s = d_1$, we obtain

$$t = \sqrt{\frac{2s}{a}} = \sqrt{\frac{4R_E}{\frac{4}{3}\pi R_E \rho_1 G}} = \sqrt{\frac{3}{\pi \rho_1 G}} \approx 1970\ s \approx 32{,}8\ min$$ (f)

b) Assume there was a similar tunnel between London and Berlin. How long would that journey take?

In this scenario, only two quantities are changed from part a): the force accelerating the train and the length of the tunnel. However, the distance r from the train to Earth's core is no longer ideally suited to describe the train's position. Instead, we consider



$$x(r) = r\cos(\alpha),$$

where α is the angle between a line perpendicular to the tunnel through the Earth's core and the direction of gravity. From the sketch we see that only the comp. of gravity acting parallel to the tunnel produces acceleration on the train, i.e.

$$F_{ac} = F_g \cdot \cos(\alpha) = -\underbrace{\frac{4}{3}\pi G \rho_1}_{k^2} m_{train}\, r \cos(\alpha) = -k^2 m_{train}\, x(r)$$ (✓)

Surprisingly, this yields the same e.o.m. as in a), just with a new coordinate, namely How?

$$m_{train}\ddot{x}(r) = F_{ac} = -k^2 m_{train}\, x(r)$$ (✓)

Since the suspension rate of this new harmonic oscillator remains the same as the old one, we can infer that its period, and hence the travel time, is also unchanged at 42.2 minutes.

c) What is the period T of a hypothetical satellite that orbits the Earth on its surface? Why is it exactly the same as the time for the return journey of the Earth?

A stable orbit requires opposing gravitational and centrifugal forces of equal strength. From this, one can calculate the orbital velocity at a given distance from the Earth, in this case R_E.

$$F_g = -G\frac{M_E m_{sat}}{R_E^2} = -\frac{m_{sat} v^2}{R_E} = -F_c \implies v = \sqrt{\frac{G M_E}{R_E}},$$

where $M_E = \int_{V_E} \rho(r)\, d^3r = \frac{4}{3}\pi R_E^3 \rho_0$. With this, the orbital period T follows easily.

$$T = \frac{s}{v} = \frac{2\pi R_E}{\sqrt{\frac{4}{3}\pi R_E^2 G\rho_0}} = \sqrt{\frac{3\pi}{G\rho_0}} = 2t_{travel}$$

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2. Lagrangian Uniqueness (10 points)

Consider the Lagrangian for a point mass within a potential $V(q)$,

$$\mathcal{L}(q,\dot{q},t) = \frac{m}{2}\dot{q}^2 - V(q). \tag{1}$$

a) Show that the equations of motion (Euler-Lagrange eqs.) won't change under transformations of the type

$$\mathcal{L}(q,\dot{q},t) \to \mathcal{L}'(q,\dot{q},t) = \alpha\mathcal{L}(q,\dot{q},t) + \frac{d}{dt}f(q,t), \tag{2}$$

or, in other words, that the action $S = \int dt\, \mathcal{L}(q,\dot{q},t)$ is invariant under scaling and gauge transformations which add only a total time derivative to the Lagrangian.

By an application of the principle of stationary action to the action functional $S[\mathcal{L}]$, one can derive the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0, \qquad \text{where } \mathcal{L} = \mathcal{L}(q(t), \dot{q}(t), t).$$

We perform this derivation explicitly for a transformed Lagrangian as defined by the exercise and show that it yields the above e.o.m.
First, we note that the action under such a transformation becomes

$$S \longrightarrow S' = \int_{t_i}^{t_f} \mathcal{L}'(q, \dot{q}, t)\, dt = \int_{t_i}^{t_f} \left(\alpha \mathcal{L}(q, \dot{q}, t) + \frac{d}{dt} f(q, t)\right) dt = \alpha S + f(q,t)\Big|_{t=t_f} - f(q,t)\Big|_{t=t_i}$$

ok

where t_i, t_f are initial and final time. Now we perform a variation of S' w.r.t. $q = q(t)$, where the endpoints, $q_i = q(t=t_i)$ and $q_f = q(t=t_f)$, are held fixed, i.e. $\delta q_i = \delta q_f = 0$.

$$\delta S' = \alpha \delta S + \underbrace{\delta f(q,t)\Big|_{t=t_f}}_{0} - \underbrace{\delta f(q,t)\Big|_{t=t_i}}_{0} = \alpha \delta S$$

Apparently, the variation of S' is simply that of S multiplied by the scaling factor. From this, we can immediately see, that the Euler-Lagrange-eqs. remain invariant up to a multiplication by the scaling factor

$$\alpha\left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q}\right) = 0,$$

which we can safely get rid of by division.

b) What is the physical interpretation of the property you just showed?
In part a), we showed that the Lagrangian is invariant under gauge transformations. The transformation we applied to the Lagrangian is equivalent to a gauge transformation of the 4-vector potential A^μ. ok?
Among other possible interpretations, gauge invariance can be seen as a manifestation of the nonobservability of A^μ.
This question more likely aims at the fact that with gauge invariance, one associates symmetries and conservation laws. ok.

c) Show that under the infinitesimal transformation $q \to q + dq$, the Lagrangian gains a contribution of potential energy that can be written in the familiar form $F ds$. This means, that in this case, we don't have homogeneity of space.

Since generalized coordinates and generalized velocities are all independent variables, transforming the Lagrangian $\mathcal{L}(q, \dot{q}, t) = \frac{m}{2}\dot{q}^2 - V(q)$ amounts to only transforming $V(q)$. Since we apply an infinitesimal transformation, we may Taylor-expand $V(q+dq)$ to linear order without picking up an error using the formula for the Taylor polynomial to order n of $f(x)$ at $x=a$,

$$T_n(f(x), a) = \sum_{k=0}^{n} \frac{1}{k!} \frac{d^k f(x)}{d^k x}\bigg|_{x=a} (x-a)^k ,$$

where in our case $n=1$, $f(x) = V(q+dq)$, $x = q+dq$ and $a = q$. We obtain

$$V(q+dq) \approx \frac{1}{0!} \frac{d^0 V(q+dq)}{d^0 (q+dq)}\bigg|_{q+dq=q} (q+dq-q)^0 + \frac{1}{1!} \frac{d^1 V(q+dq)}{d^1 (q+dq)}\bigg|_{q+dq=q} (q+dq-q)^1$$

$$= V(q) + V'(q)\, dq$$

Hence, the Lagrangian changes as

$$\mathcal{L} \xrightarrow{q \to q+dq} \mathcal{L}' = \frac{m}{2}\dot{q}^2 - V(q) - V'(q)\, dq ,$$

where $-V'(q) dq = F(q) dq$ in analogy to $F ds$, where $F(q)$ is a conservative force.

d) Give an example of a symmetry that the above Lagrangian $\mathcal{L}$ does exhibit, and name the associated conserved quantity.

The above Lagrangian has several symmetries. For instance, it remains invariant under the transformation $\dot{q} \to -\dot{q}$. However, there is no assoc. conserved quantity since this is not a continuous symmetry.

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By contrast, $t \to t' = t + \varepsilon$ is a continuous symmetry. For clarity, q and $\dot{q}$ are independent and do not participate in the transformation, i.e. $q(t) \to q(t)$, $\dot{q}(t) \to \dot{q}(t)$.

The conserved quantity associated with time translational invariance is ~~momentum~~ energy sorry :D

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3. Newton's Space Rope (10 points)

Consider a rope that is connected to the surface of the Earth on the equator. It is affected by the gravitational force

$$F_g(r) = G\frac{M\,m}{r^2},$$

as well as a centrifugal force

$$F_c(r) = \frac{m v^2}{r} = m\,\omega^2 r.$$

a) Write the forces as integrals along the rope axis. Assume a constant length density σ.

This seems pointless. For points, we still note

$$F_g(\overline{R_E H}) = \int_{R_E}^{H} G\frac{M}{r^2}\,\sigma\,dr\,, \qquad F_c(\overline{R_E H}) = \int_{R_E}^{H} \sigma\,\omega^2 r\,dr,$$

where H denotes the height of the space rope from Earth's core and $F_g(\overline{R_E H})$ stands for the accumulated force along the entire rope $\overline{R_E H}$.

b) At what point do the two forces cancel out, i.e. how long does the rope have to be to be in equilibrium? Use the dimensionless variable $\xi = \frac{x}{R_E}$ to express its length in Earth radii.

At Earth's geostationary orbital radius R_g, gravitational and centrifugal forces cancel, i.e.

$$F_g(R_g) = -G\frac{M\,m}{R_g^2} = -m\,\omega^2 R_g = -F_c(R_g).$$

From this, we gather that $\omega^2 = \frac{GM}{R_g^3}$. For the space rope to be in equilibrium, we now impose

$$0 \stackrel{!}{=} F_g(\overline{H R_E}) + F_c(\overline{H R_E}) = \int_{R_E}^{H} G\frac{M}{r^2}\,\sigma\,dr + \int_{R_E}^{H} \sigma\,\frac{GM}{R_g^3}\,r\,dr = GM\sigma\int_{R_E}^{H}\left(\frac{1}{r^2} + \frac{r}{R_g^3}\right)dr$$

$$= GM\sigma\left(-\frac{1}{H} + \frac{1}{R_E} + \frac{1}{2R_g^3}\left(H^2 - R_E^2\right)\right)$$

The obtained expression is a cubic polynomial which we need to solve for H.

We drop the prefactor and rewrite:

$$0 = H - R_E + \frac{R_E H}{2R_J^3}\left(H^2 - 2R_E H + R_E^2 + 2R_E H - 2R_E^2\right) = H - R_E + \frac{R_E H}{2R_J^3}\left((H-R_E)^2 + 2R_E(H-R_E)\right)$$

$$= (H - R_E)\left(1 + \frac{R_E H}{2R_J^3}(H - R_E + 2R_E)\right)$$

Apparently, $(H - R_E)$ can be factorised from our polynomial meaning $H = R_E$ is a solution. However, it is obviously not the one we are looking for.

droids

We turn towards the second factor

$$0 = 1 + \frac{R_E H}{2R_J^3}(H + R_E) \quad \Longleftrightarrow \quad 0 = R_E\left(H^2 + R_E H + \frac{2R_J^3}{R_E}\right) = R_E\left(\left(H + \frac{R_E}{2}\right)^2 - \frac{R_E^2}{4} + \frac{2R_J^3}{R_E}\right)$$

The remaining factor yields two solutions one of which we discard due to negativity.

$$H = -\frac{R_E}{2} \underset{(-)}{+} \sqrt{\frac{R_E^2}{4} - \frac{2R_J^3}{R_E}}, \qquad x = H - R_E = -\frac{3R_E}{2} + \sqrt{\frac{R_E^2}{4} - \frac{2R_J^3}{R_E}} \approx 148{,}800 \text{ km}, \quad \xi \approx 22.58$$

good

c) What is the orbital velocity of the outermost point of the rope? (You may assume that the rope is rigid)

The velocity follows easily from

$$v = \frac{s}{t} = \frac{2\pi H}{d} \approx 10.92 \frac{\text{km}}{\text{s}} \qquad (d \text{ stands for one Earth day})$$

$\frac{10}{10}$

4. Keplerian Orbits (10 points)

Given the Lagrangian

$$\mathcal{L} = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - U(r), \tag{3}$$

we're going to find out whether Keplerian orbits are closed.

a) Name q and $\dot{q}$. What assumption about the shape of the orbits has been made? Is this a sensible choice given what you know about planetary orbits?

$~~q = r, \quad \dot{q} = \dot{r}^2 + r^2\dot{\varphi}^2~~$

$$q = \{r, \varphi\}$$
$$\dot{q} = \{\dot{r}, \dot{\varphi}\}$$

The Lagrangian in eq. (3) assumes that there is no ~~orbital movement~~ along the polar angle, ~~i.e. $\dot{\vartheta} = 0$~~. ! For this to be true requires all orbital motion to lie in a two-dimensional plane. Based on our knowledge of the orbits of the planets in our solar system, this is a reasonable assumption.

b) What is the physical interpretation of the three summands?

Assumingly, this question refers to the three summands in Lagrangian (3). The first two combined represent the kinetic energy. The third is the potential energy.

+ rotational $\mathcal{E}$. !

c) What is the total energy $\mathcal{E}$ of the system? Where applicable, write it in terms of the canonical momentum in φ-direction $p_\varphi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}$.

p_φ can be easily calculated

$$p_\varphi = m r^2 \dot{\varphi}$$

$\mathcal{E}$ is simply the sum rather than the subtraction of kinetic and potential energy in the Lagrangian, i.e.

$$\mathcal{E} = \frac{m}{2}\left(\dot{r}^2 + r^2 \dot{\varphi}^2\right) + U(r) = \frac{m}{2}\dot{r}^2 + \frac{p_\varphi^2}{2mr^2} + U(r)$$

d) From the above, you can easily arrive at

$$\frac{dr}{dt} = \sqrt{\frac{2}{m}\left(\mathcal{E} - U(r)\right) - \frac{p_\varphi^2}{2m^2r^2}} \qquad (4)$$

Try to express $d\varphi$ in terms of dr! (Tip: $p_\varphi = mr^2 \frac{d\varphi}{dt}$). Extra question: What condition needs to apply that φ describes a closed curve?

Vague wording. We assume here that $d\varphi$ does not denote the complete differential since $\mathcal{L}$ is cyclic in φ. Perhaps

$$\frac{d\varphi}{dt} = \sqrt{\frac{2}{mr^2}\left(\mathcal{E} - U(r)\right) - \underbrace{\frac{1}{r^2}\left(\frac{dr}{dt}\right)^2}_{\frac{p_\varphi^2}{m^2r^4}}}$$

is what is asked for. Extra answer: $\dot{r} \stackrel{!}{=} 0$.

no

e) From eq. (4), we can immediately see that $\dot{r}=0$ iff

$$\mathcal{E} = U(r) + \frac{p_\varphi^2}{2m^2r^2} = U(r) + \frac{m}{2}r^2\dot{\varphi}^2.$$

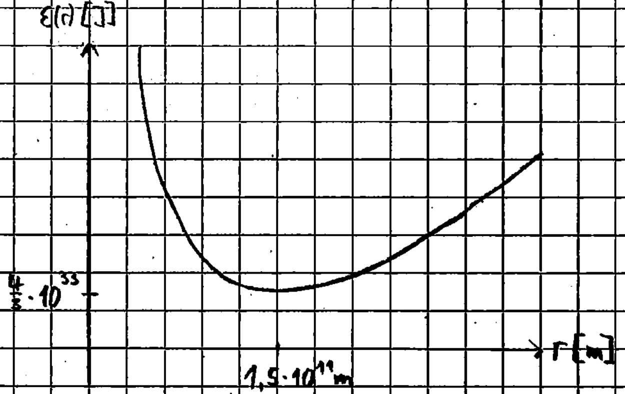
What very special form of orbit do we have in the case of $\dot{r}=0$?
Given $U(r) = G\frac{Mm}{r}$, produce a plot of $\mathcal{E}(r)$ for

$$M = 2\cdot 10^{30}\,kg,\quad m = 10^{24}\,kg,\quad \dot{\varphi} = 2\cdot 10^{-7}\,\frac{1}{s}.$$

If done correctly, you should see a minimum at $r \approx 150\cdot 10^6\,km$. Guess whose potential you just plotted!

For $\dot{r}=0$, our orbit is a circle. We guess its Jupiter's potential!!

no. Earth!



Real plot pls thx :)

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5. Extra: Mechanical Similarity (5 points)

A team of astronauts lands on Mars. They transmit their first steps on the red planet live on TV and in order to prove that it's not a hoax, one of them uses a simple pendulum of length $l = 1\,m$ to show the audience that they really are on Mars.

As the gravitational constant is supposedly lower on Mars than in a TV studio on Earth, the pendulum should have a larger period $T \propto \sqrt{\frac{l}{g}}$. Can you, as TV audience, distinguish between a lower gravitational constant g and a slower passage of time? Could this still be hoax?

No, we can't. It could still be a hoax. Reason?

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