

Group Theory - Problem Sheet 1

Exercise 1 (Definitions)

a) Define what a group is and show from the axioms that left-identity and -inverses are also right-identity and -inverses, and that identity and inverses are unique.

group: algebraic structure $(G, \langle \cdot, \cdot \rangle)$ consisting of

- a set of elements G and
- an operation $\langle \cdot, \cdot \rangle$ that satisfies the

group axioms

1. Closure: $\forall a, b \in G \langle a, b \rangle \in G$

2. Associativity $\forall a, b, c \in G \langle \langle a, b \rangle, c \rangle = \langle a, \langle b, c \rangle \rangle$

3. Identity $\exists e \in G \forall a \in G: \langle e, a \rangle = a$

4. Inverse $\exists a^{-1} \in G \forall a \in G: \langle a^{-1}, a \rangle = e$

From now on, the group operation between two elements of a group is implicit, i.e. $\langle a, b \rangle \rightarrow ab$. We prove that left- and right-identity and inverse are equal and their uniqueness:

$$a^{-1}a = e \Rightarrow a^{-1}(a a^{-1}) = \underbrace{(a^{-1}a)}_e a^{-1} = e a^{-1} = a^{-1} \Rightarrow a a^{-1} = e \text{ (right inverse)}$$

$$a a^{-1} = e \Rightarrow a e = a(a^{-1}a) = \underbrace{(a a^{-1})}_e a = e a \Rightarrow e a = a = e a \text{ (right identity)}$$

$$\text{assume } \exists e, e' \in G: e a = a, e' a = a \forall a \in G \Rightarrow e = e e' = e' \Rightarrow e = e' \text{ (identity uni.)}$$

$$\text{assume } \exists a^{-1}, a'^{-1} \in G: a^{-1}a = e, a'^{-1}a = e \forall a \in G \Rightarrow a^{-1}a = e = a'^{-1}a \Rightarrow a^{-1} = a'^{-1} \text{ (inverse uni.)}$$

b) What is a group homomorphism? Show that the map $\mathbb{Z} \rightarrow \mathbb{Z}_n, z \mapsto [z]$ is a group homomorphism.

group homomorphism: given two groups $(G, *)$, $(H, \cdot)$, $\varphi: G \rightarrow H$ is a group homomorphism if $\varphi(g_1 * g_2) = \varphi(g_1) \cdot \varphi(g_2) \quad \forall g_1, g_2 \in G$

$$\forall z_1, z_2 \in \mathbb{Z}: [z_1 + z_2] = [z_1] + [z_2]$$

c) What is a subgroup? Show that the rotations form a subgroup $\mathbb{Z}_n \subset D_n$ of the dihedral group D_n .

given two groups $(G, *)$, $(H, \cdot)$, $(G, *)$ is a subgroup of $(H, \cdot)$ if $\forall g \in G, g \in H$

$D_n = \{R_0, R_1, \dots, R_{n-1}, S_0, S_1, \dots, S_{n-1}\}$, the group of symmetries of a regular n -gon

$$\mathbb{Z}_n = D_n / \{S_0, S_1, \dots, S_{n-1}\} \subset D_n$$

d) What are conjugacy classes? Determine all conjugacy classes of D_4 and D_6 .
What about general D_n ?

$$\text{conjugacy class} = [g] = \{g' \in G \mid \exists g' g g'^{-1} = g\}$$

$$D_4 = \{R_0, R_1, R_2, R_3, S_0, S_1, S_2, S_3\}$$

$$[R_0] = \{R_0, R_1, R_2, R_3, S_0, S_1, S_2, S_3\}$$

Exercise 2 (Finite groups)

- a) Let G be a finite group with a prime number of elements. What can you say about G ?

By Lagrange's theorem, G can only have the two trivial subgroups $\{e\}$ and G . Since we can always construct orbits Γ_g for all elements $g \in G$, which by the closure axiom are subgroups of G , we have $\Gamma_e = \{e\}$ or $\Gamma_g = G$. Since Γ_g is isomorphic to $\mathbb{Z}_m$, $\Gamma_g \cong \mathbb{Z}_m$, we have that G is isomorphic to $\mathbb{Z}_m$, $G \cong \mathbb{Z}_m$.

- b) Write the multiplication table of $\mathbb{Z}_3$ and the respective embedding $\mathbb{Z}_3 \hookrightarrow S_3$ into the symmetric group.

$\mathbb{Z}_3$	0	1	2	$\mathbb{Z}_3 \hookrightarrow S_3$
0	0	1	2	$\pi_0: 0 \rightarrow \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}$
1	1	2	0	$\pi_1: 1 \rightarrow \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$
2	2	0	1	$\pi_2: 2 \rightarrow \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$

Exercise 3 (Symmetric group)

- a) Write the permutation $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 5 & 2 & 4 & 6 \end{pmatrix}$ as product of non-intersecting cycles.

$$\pi = (1 \ 3 \ 5 \ 4 \ 2)(6)$$

- b) What is the order of S_3 ? How many conjugacy classes does S_3 have? Determine their sizes, and give a representation of each one.

$|S_3| = 3! = 6$. S_3 has $p(3) = 3$ conjugacy classes, where $p(n)$ is the number of partitions of n . Conjugacy class sizes can be calculated using $|C_{(k_1, \dots, k_l)}| = \frac{n!}{i_1! i_2! \dots i_l!}$, where the k_i give the number of permutation cycles of length i .

Considering the different partitions of n , we get

$$\lambda = (3, 0, 0), \quad k = (3, 0, 0)$$

$$c_{(\lambda, \alpha)} = \frac{3!}{1^3 \cdot 3! \cdot 2^0 \cdot 0! \cdot 3^0 \cdot 0!} = 1$$

$$\lambda = (2, 1, 0), \quad k = (1, 1, 0)$$

$$c_{(\lambda, \alpha)} = \frac{3!}{1^1 \cdot 1! \cdot 2^1 \cdot 1! \cdot 3^0 \cdot 1!} = 3$$

$$\lambda = (1, 1, 1), \quad k = (0, 2, 1)$$

$$c_{(\lambda, \alpha)} = \frac{3!}{1^1 \cdot 0! \cdot 2^2 \cdot 0! \cdot 3^1 \cdot 1!} = 2$$

Representatives: $\lambda = (3, 0, 0): e = (1)(2)(3)$

$$\lambda = (2, 1, 0): (12)(3)$$

$$\lambda = (1, 1, 1): (123)$$

c) Show that $H = \{e, (1, 2)\} \subset S_3$ is a subgroup. Is it normal?

$$\text{Closure of } H: \quad ee = e \in H, \quad e(1, 2) = (1, 2) \in H, \quad (1, 2)e = (1, 2) \in H$$

$$(1, 2)(1, 2) = e \in H$$

Associativity: Is inherited from S_3 , since $e, (1, 2) \in S_3$.

Identity: $e \checkmark$

Inverse: $ee = e, \quad (1, 2)(1, 2) = e \checkmark$

All four group axioms are met. Therefore $H \subset S_3$ is a subgroup.

Since H cannot be formed by conjugacy classes of S_3 , it is not normal in S_3 .

Exercise 4 (Commutator subgroup)

Let G be a group. The commutator of two elements $g, h \in G$ is defined by $[g, h] = g^{-1}h^{-1}gh \in G$. The commutator-subgroup of G is the group generated by all commutators in G :

$$[G, G] = \{[g_1, h_1], \dots, [g_n, h_n] : n \in \mathbb{N}, g_i, h_i \in G\}$$

a) Show that the commutator subgroup is indeed a subgroup $[G, G] \subset G$.

We use the condition that if for all $[g, h], [k, l] \in [G, G]$, $[g, h][k, l]^{-1} \in [G, G]$, where $g, h, k, l \in G$, then $[G, G] \trianglelefteq G$ is a subgroup.

We have $[k, l]^{-1} = (k^{-1}l^{-1}k)^{-1} = l^{-1}k^{-1}lk = [l, k]$ and therefore $[g, h][k, l]^{-1} = [g, h][l, k] \in [G, G]$ by definition of $[G, G]$.

b) Show that it is a normal subgroup $[G, G] \trianglelefteq G$.

$[G, G] \trianglelefteq G$ is a normal subgroup $\Leftrightarrow \forall g \in G: g[G, G]g^{-1} = [G, G]$

We prove the above equality by showing that each is a subgroup of the other.

$g[G, G]g^{-1} \subseteq [G, G]:$

$\forall g \in G, [k, l] \in [G, G]$, we have $[k, l]^{-1}g[k, l]g^{-1} = [[k, l], g^{-1}] \in [G, G]$.

From this, it follows that, since $[G, G] \trianglelefteq G$ is a subgroup fulfilling closure w.r.t. the group action,

$$[k, l][k, l]^{-1}g[k, l]g^{-1} = g[k, l]g^{-1} \in [G, G]$$

$g[G, G]g^{-1} \supseteq [G, G]:$

$\forall [k, l] \in [G, G] \exists m, n \in G: gmg^{-1} = k \wedge gng^{-1} = l$,

where we used that conjugation is a homomorphism on G

$$\begin{aligned} \Rightarrow [k, l] &= [gmg^{-1}, gng^{-1}] = (gmg^{-1})^{-1}(gng^{-1})^{-1}gmg^{-1}gng^{-1} \\ &= g m^{-1} g^{-1} g n^{-1} g^{-1} g m g^{-1} g n g^{-1} = g m^{-1} n^{-1} m n g^{-1} \\ &= g[m, n]g^{-1} \in g[G, G]g^{-1} \end{aligned}$$

Therefore, $[G, G] \trianglelefteq G$.

Exercise 5 (Group actions)

An action of a group G on a set X is a group homomorphism $\varphi: G \rightarrow \text{Bij}(X)$ from G into the group $\text{Bij}(X)$ of bijections from X to X . One often uses the abbreviated notation $gx := \varphi(g)(x)$. For any $x \in X$ its orbit in G is defined by

$$\text{Orb}_G(x) = Gx = \{gx \mid g \in G\} \subset X.$$

The stabilizer of $x \in X$ is defined by

$$\text{Stab}_G(x) = \{g \in G \mid gx = x\} \subset G.$$

a) Show that the stabilizer is a subgroup of G .

For this, we again resort to the 'one-step' subgroup test, i.e. we show that $\text{Stab}_G(x) \subset G$ is a subset of G and that $\text{Stab}_G(x) \neq \emptyset$ is not empty (both of which is fulfilled by definition) and that $\forall a, b \in \text{Stab}_G(x)$, we have $ab^{-1} \in \text{Stab}_G(x)$, where always $x \in X$.

For such a and b , we can write

$$b^{-1}x = b^{-1}(bx) \stackrel{!}{=} (b^{-1}b)x = x \quad \text{where we used that } \varphi \text{ is homomorphism}$$

$$\Rightarrow (ab^{-1})x = a(b^{-1}x) = ax = x$$

$$\Rightarrow ab^{-1} \in \text{Stab}_G(x) \quad \text{and} \quad \text{Stab}_G(x) \subset G$$

b) Show that stabilizers of elements $x, y' \in X$ which lie in the same G -orbit (i.e. $y, y' \in \text{Orb}_G(x)$ for some $x \in X$) are conjugate.

Since $x, y' \in \text{Orb}_G(x)$, $\exists a, b \in G: ax = y, bx = y'$. G being a group, we can invert to get $a^{-1}y = x = b^{-1}y'$.

For any $g \in \text{Stab}_G(y)$, we have $gy = y \Leftrightarrow g(ax) = ax \Leftrightarrow a^{-1}g(ax) = x$.

Inserting $x = b^{-1}y'$ gives $a^{-1}g(ab^{-1}y') = b^{-1}y' \Leftrightarrow (ba^{-1})g(ab^{-1}y') = y'$, from which we deduce that $(ba^{-1})g(ab^{-1}) =: hgh^{-1} \in \text{Stab}_G(y')$.

Since this holds for all $g \in \text{Stab}_G(y)$, we found $h\text{Stab}_G(y)h^{-1} = \text{Stab}_G(y')$.

c) Show that the map $Gx \rightarrow G/\text{Stab}_G(x)$, which sends
 $gx \mapsto g \text{Stab}_G(x)$ is well defined (i.e. $gx = hx \Rightarrow g \text{Stab}_G(x) = h \text{Stab}_G(x)$)
and is a bijection.