

String Theory

Solution to Assignment 1

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1 Differential Geometry for General Relativity

Consider the line element of a 2-sphere of radius a ,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = a^2 [d\theta^2 + \sin^2(\theta) d\phi^2]. \quad (1)$$

The metric $g_{\mu\nu}$ encodes all information on the geometry of a manifold. From it one can determine all those geometric quantities that are relevant for general relativity, namely

The metric Choosing $x^1 = \theta$ and $x^2 = \phi$, read off the matrix $g_{\mu\nu}$.

The Christoffel symbols The Christoffel symbols are defined as

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa} \left(\frac{\partial g_{\mu\kappa}}{\partial x^\nu} + \frac{\partial g_{\nu\kappa}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\kappa} \right). \quad (2)$$

They enter the covariant derivatives $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\lambda_{\mu\nu} V^\lambda$, where the correction term with the Christoffel symbol ensures that the covariant derivative indeed transforms covariantly under arbitrary coordinate transformations $x^\mu \rightarrow x'^\mu(x^\nu)$, i.e.

$$\nabla_\mu V^\nu = \partial_\mu V^\nu \rightarrow (\nabla_\mu V^\nu = \partial_\mu V^\nu)' = \frac{\partial x^\lambda}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\rho} \nabla_\lambda V^\rho, \quad (3)$$

without second derivatives in the coordinates. Compute the non-vanishing Christoffel symbols for the 2-sphere (Hint: $\Gamma^\kappa_{\lambda\mu} = \Gamma^\kappa_{\mu\lambda}$).

The Riemann tensor The Riemann curvature tensor has the form

$$R^\kappa_{\lambda\mu\nu} = \partial_\mu \Gamma^\kappa_{\lambda\nu} - \partial_\nu \Gamma^\kappa_{\lambda\mu} + \Gamma^\eta_{\lambda\nu} \Gamma^\kappa_{\eta\mu} - \Gamma^\eta_{\lambda\mu} \Gamma^\kappa_{\eta\nu}. \quad (4)$$

Calculate the non-vanishing components of the Riemann tensor for the 2-sphere (Hint: $R^\kappa_{\lambda\mu\nu} = -R^\kappa_{\lambda\nu\mu}$).

Remark: The Riemann tensor measures the curvature of a space, for instance by quantifying the non-commutativity of the covariant derivatives,

$$[\nabla_\mu, \nabla_\nu] V^\kappa = R^\kappa_{\lambda\mu\nu} V^\lambda. \quad (5)$$

A space with vanishing Riemann tensor is flat, i.e. the metric can be brought to the standard Minkowskian (or Euclidean) form by means of a coordinate transformation.

The Ricci tensor The Ricci tensor is defined as

$$R_{\mu\nu} = R^\kappa_{\mu\kappa\nu}. \quad (6)$$

Calculate the Ricci tensor for the 2-sphere.

The scalar curvature The scalar curvature is given by

$$\mathcal{R} = g^{\mu\nu} R_{\mu\nu}. \quad (7)$$

Calculate the scalar curvature of the 2-sphere. How does it behave in the limit $a \rightarrow \infty$? Interpret this behavior.

The Einstein tensor The Einstein tensor appears in the field equation of general relativity and it relates the curvature of space-time to the matter distribution,

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \quad (8)$$

where G denotes Newton's constant, $T_{\mu\nu}$ is the energy-momentum tensor, and $G_{\mu\nu}$ denotes the Einstein tensor,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu}. \quad (9)$$

Calculate the Einstein tensor for the 2-sphere.

The metric For $x^1 = \theta$ and $x^2 = \phi$, the equation

$$g_{\mu\nu}dx^\mu dx^\nu = a^2[d\theta^2 + \sin^2(\theta)d\phi^2] \quad (10)$$

implies

$$\mathbf{g} = a^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin^2(\theta) \end{pmatrix}, \quad \text{and hence} \quad \mathbf{g}^{-1} = \frac{1}{a^2} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2(\theta)} \end{pmatrix}. \quad (11)$$

The Christoffel symbols Since the Christoffel symbols carry three coordinate indices and we have $d = 2$ dimensions (θ, ϕ) , there are $d^3 = 8$ Christoffel symbols in total. However, due to the symmetry in the lower two indices, those with lower indices 12 and 21 are equal both for an upper index of 1 and 2, so only $8 - 2 = 6$ of those symbols are independent. We calculate each of those in turn. Since the only nonvanishing metric derivative is $\frac{\partial g_{22}}{\partial x^1} = 2\sin(\theta)\cos(\theta)$, all but Γ^1_{22} , Γ^2_{12} , and Γ^2_{21} can immediately be seen to vanish:

$$\Gamma^1_{11} = \frac{1}{2}g^{1\kappa} \left(\frac{\partial g_{1\kappa}}{\partial x^1} + \frac{\partial g_{1\kappa}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^\kappa} \right) = 0, \quad (12)$$

$$\Gamma^1_{12} = \frac{1}{2}g^{1\kappa} \left(\frac{\partial g_{1\kappa}}{\partial x^2} + \frac{\partial g_{2\kappa}}{\partial x^1} - \frac{\partial g_{12}}{\partial x^\kappa} \right) = \frac{1}{2}g^{12} \frac{\partial g_{22}}{\partial x^1} = 0 = \Gamma^1_{21}, \quad (13)$$

$$\begin{aligned} \Gamma^1_{22} &= \frac{1}{2}g^{1\kappa} \left(\frac{\partial g_{2\kappa}}{\partial x^2} + \frac{\partial g_{2\kappa}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^\kappa} \right) \\ &= -\frac{1}{2}g^{11} \frac{\partial g_{22}}{\partial x^1} = -\frac{1}{2a^2} \frac{\partial [a^2 \sin^2(\theta)]}{\partial \theta} = -\sin(\theta)\cos(\theta), \end{aligned} \quad (14)$$

$$\Gamma^2_{11} = \frac{1}{2}g^{2\kappa} \left(\frac{\partial g_{1\kappa}}{\partial x^1} + \frac{\partial g_{1\kappa}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^\kappa} \right) = 0, \quad (15)$$

$$\begin{aligned}\Gamma_{12}^2 &= \frac{1}{2}g^{2\kappa} \left(\frac{\partial g_{1\kappa}}{\partial x^2} + \frac{\partial g_{2\kappa}}{\partial x^1} - \frac{\partial g_{12}}{\partial x^\kappa} \right) \\ &= \frac{1}{2}g^{22} \frac{\partial g_{22}}{\partial x^1} = \frac{1}{2a^2 \sin^2(\theta)} \frac{\partial [a^2 \sin^2(\theta)]}{\partial \theta} = \cot(\theta) = \Gamma_{21}^2,\end{aligned}\tag{16}$$

$$\Gamma_{22}^2 = \frac{1}{2}g^{2\kappa} \left(\frac{\partial g_{2\kappa}}{\partial x^2} + \frac{\partial g_{2\kappa}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^\kappa} \right) = 0.\tag{17}$$

The Riemann tensor The Riemann tensor has four indices. So for $d = 2$ dimensions, the tensor contains a total of $d^4 = 16$ components. Due to the antisymmetry $R^\kappa_{\lambda\mu\nu} = -R^\kappa_{\lambda\nu\mu}$ of the last two indices, there is only $\frac{d}{2}(d-1) = 1$ independent combination of those two indices, leaving the Riemann tensor with $d^2 \cdot \frac{d}{2}(d-1) = 4$ independent components. In particular, all eight entries where the last two indices are equal must be zero, i.e.

$$R^1_{111} = R^1_{211} = R^2_{111} = R^2_{211} = 0,\tag{18}$$

$$R^1_{122} = R^1_{222} = R^2_{122} = R^2_{222} = 0.\tag{19}$$

The remaining eight components are potentially nonzero, but form four pairs of two whose members differ only in sign. These we calculate by hand:

$$R^1_{112} = -R^1_{121} = \partial_1 \Gamma^1_{12} - \partial_2 \Gamma^1_{11} + \Gamma^\eta_{12} \Gamma^1_{\eta 1} - \Gamma^\eta_{11} \Gamma^1_{2\eta} = 0,\tag{20}$$

$$\begin{aligned}R^1_{212} &= -R^1_{221} = \partial_1 \Gamma^1_{22} - \partial_2 \Gamma^1_{21} + \Gamma^\eta_{12} \Gamma^1_{\eta 1} - \Gamma^\eta_{21} \Gamma^1_{2\eta} = \partial_1 \Gamma^1_{22} - \Gamma^2_{21} \Gamma^1_{22} \\ &= \partial_\theta [-\sin(\theta) \cos(\theta)] - \cot(\theta) \cdot [-\sin(\theta) \cos(\theta)] \\ &= -\cos^2(\theta) + \sin^2(\theta) + \cos^2(\theta) = \sin^2(\theta),\end{aligned}\tag{21}$$

$$\begin{aligned}R^2_{112} &= -R^2_{121} = \partial_1 \Gamma^2_{12} - \partial_2 \Gamma^2_{11} + \Gamma^\eta_{12} \Gamma^2_{\eta 1} - \Gamma^\eta_{11} \Gamma^2_{2\eta} = \partial_1 \Gamma^2_{12} + \Gamma^2_{12} \Gamma^2_{21} \\ &= \partial_\theta \cot(\theta) + \cot(\theta) \cdot \cot(\theta) = -1 - \cot^2(\theta) + \cot^2(\theta) = -1,\end{aligned}\tag{22}$$

$$R^2_{212} = -R^2_{221} = \partial_1 \Gamma^2_{22} - \partial_2 \Gamma^2_{21} + \Gamma^\eta_{22} \Gamma^2_{\eta 1} - \Gamma^\eta_{21} \Gamma^2_{2\eta} = 0.\tag{23}$$

We found four nonvanishing components. The remark given in the exercise that a space with vanishing Riemann tensor is flat is in fact an “iff”-statement, i.e. a nonvanishing Riemann tensor implies that space is curved. We have therefore proven the unremarkable statement that the 2-sphere is curved.

The Ricci tensor The $d^2 = 4$ components of the Ricci tensor of the 2-sphere are given by

$$R_{11} = R^\kappa_{1\kappa 1} = R^1_{111} + R^2_{121} = 1,\tag{24}$$

$$R_{12} = R^\kappa_{1\kappa 2} = R^1_{112} + R^2_{122} = 0,\tag{25}$$

$$R_{21} = R^\kappa_{2\kappa 1} = R^1_{211} + R^2_{221} = 0,\tag{26}$$

$$R_{22} = R^\kappa_{2\kappa 2} = R^1_{212} + R^2_{222} = \sin^2(\theta).\tag{27}$$

The scalar curvature In the case of the 2-sphere, $\mathcal{R}$ takes the very simple and memorable form,

$$\mathcal{R} = g^{\mu\nu} R_{\mu\nu} = g^{11} R_{11} + g^{22} R_{22} = \frac{1}{a^2} \cdot 1 + \frac{1}{a^2 \sin^2(\theta)} \cdot \sin^2(\theta) = \frac{2}{a^2}.\tag{28}$$

As expected, in the limit $a \rightarrow \infty$, we have

$$\lim_{a \rightarrow \infty} \mathcal{R} = 0,\tag{29}$$

i.e. a sphere of infinite radius has vanishing curvature.

The Einstein tensor We give the Einstein tensor of the 2-sphere not component-wise, but in covariant matrix form:

$$\mathbf{G} = \mathbf{R} - \frac{1}{2} \mathcal{R} \mathbf{g} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2(\theta) \end{pmatrix} - \frac{1}{2} \frac{2}{a^2} a^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin^2(\theta) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.\tag{30}$$

According to the Einstein equation,

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \quad (31)$$

a vanishing Einstein tensor requires a trivial matter distribution, $T_{\mu\nu} = 0$.

2 Transformation of tensors and tensor densities

Consider the coordinate change

$$x^\mu \rightarrow x'^\mu \equiv x'^\mu. \quad (32)$$

The associated transformation matrix and its inverse are

$$P^\mu_{\nu'} = \frac{\partial x^\mu}{\partial x'^{\nu'}}, \quad \text{and} \quad P^{\mu'}_{\nu} = \frac{\partial x'^{\mu'}}{\partial x^\nu}, \quad (33)$$

respectively. Recall that a tensor of type, say T^μ_{ν} transforms under eq. (32) as

$$T^\mu_{\nu} \rightarrow T^{\mu'}_{\nu'} = P^{\mu'}_{\alpha} P^\beta_{\nu'} T^\alpha_{\beta}. \quad (34)$$

A tensor density $\tilde{T}^\mu_{\nu}$ of weight w is defined by the transformation behavior

$$\tilde{T}^\mu_{\nu} \rightarrow \tilde{T}^{\mu'}_{\nu'} = J^w P^{\mu'}_{\alpha} P^\beta_{\nu'} \tilde{T}^\alpha_{\beta}. \quad (35)$$

(and obvious generalisations for general types of tensor densities), where $J = \det(\mathbf{P})$.

- a) Given the tensor $S_{\mu\nu}$, convince yourself that $\sqrt{\det(\mathbf{S})}$ is a scalar density of weight 1.
- b) Consider now fields of tensors and tensor densities, e.g. $T^\mu_{\nu}(x)$. Locally, i.e. infinitesimally, the transformation of eq. (32) can be parametrized as $x'^\mu = x^\mu - \epsilon^\mu(x)$. Show the following infinitesimal variations for a scalar field $\Phi(x)$, the metric $g_{\mu\nu}(x)$ and the associated metric density $\sqrt{-\det(\mathbf{g})}$:
 - i) $\delta\Phi = \epsilon^\mu \partial_\mu \Phi$,
 - ii) $\delta g_{\mu\nu} = \epsilon^\lambda \partial_\lambda g_{\mu\nu} + (\partial_\mu \epsilon^\lambda) g_{\lambda\nu} + (\partial_\nu \epsilon^\lambda) g_{\mu\lambda} = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu$,
 - iii) $\delta \sqrt{-\det(\mathbf{g})} = \partial_\lambda [\epsilon^\lambda \sqrt{-\det(\mathbf{g})}]$,

where the second equality in ii) is true for the metric connection satisfying $\nabla_\lambda g_{\mu\nu} = 0$.

Hint: For a scalar field the transformed object is defined via the relation $\Phi'(x') = \Phi(x)$.

- a) Since $S_{\mu\nu}$ is said to be a tensor, we know it transforms as

$$S_{\mu\nu} \rightarrow S_{\mu'\nu'} = P^\alpha_{\mu'} P^\beta_{\nu'} S_{\alpha\beta}. \quad (36)$$

or in matrix notation

$$\mathbf{S} \rightarrow \mathbf{S}' = \mathbf{P}_{x \rightarrow x'}^2 \mathbf{S}. \quad (37)$$

Using that the determinant of a product of matrices is the product of the determinants, we have

$$\begin{aligned} \sqrt{\det(\mathbf{S})} &\rightarrow \sqrt{\det(\mathbf{S}')} = \sqrt{\det(\mathbf{P}_{x \rightarrow x'}^2 \mathbf{S})} = \sqrt{[\det(\mathbf{P}_{x \rightarrow x'})]^2 \det(\mathbf{S})} \\ &= \det(\mathbf{P}_{x \rightarrow x'}) \sqrt{\det(\mathbf{S})} = J^1_{x \rightarrow x'} \sqrt{\det(\mathbf{S})}, \end{aligned} \quad (38)$$

and thus $\sqrt{\det(\mathbf{S})}$ is a tensor density of weight 1.

- b) We now derive the infinitesimal transformation behavior of a scalar field as well as the metric and the square root of its negated determinant under the transformation $x'^\mu = x^\mu - \epsilon^\mu(x')$.

- i) Using $x^\mu = x'^\mu + \epsilon^\mu(x')$, the variation of a **scalar field** follows from the hint that $\Phi'(x') = \Phi(x)$ together with a simple Taylor expansion of $\Phi(x)$ around x' ,

$$\begin{aligned}\Phi(x) &\rightarrow \Phi'(x') \stackrel{!}{=} \Phi(x) = \Phi(x' + \epsilon(x')) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{\partial^n \Phi(x' + \epsilon(x'))}{\partial^n (x'^\mu + \epsilon^\mu(x'))} \right|_{x'^\mu + \epsilon^\mu(x') = x'^\mu} (x'^\mu + \epsilon^\mu(x') - x'^\mu)^n \\ &= \Phi(x') + \epsilon^\mu(x') \partial_\mu \Phi(x') + \mathcal{O}[\epsilon^2(x')].\end{aligned}\quad (39)$$

From this transformation law it follows that

$$\Phi'(x) = \Phi(x) + \epsilon^\mu(x) \partial_\mu \Phi(x) + \mathcal{O}[\epsilon^2(x)], \quad (40)$$

and hence the variation $\delta\Phi(x)$ of the scalar field $\Phi(x)$ given by the difference of the transformed and the original field reads

$$\delta\Phi(x) = \Phi'(x) - \Phi(x) = \epsilon^\mu(x) \partial_\mu \Phi(x) + \mathcal{O}[\epsilon^2(x)]. \quad (41)$$

- ii) For the **metric** $g_{\mu\nu}(x)$, we know that it strictly follows the transformational behavior (34) of a tensor (field). Therefore,

$$g_{\mu\nu}(x) \rightarrow g_{\mu'\nu'}(x') = P^\alpha_{\mu'} P^\beta_{\nu'} g_{\alpha\beta}(x). \quad (42)$$

Now, all we have to do is expand the expression on the right to first order in $\epsilon^\mu(x)$. This can be done by inserting our transformation $x^\mu = x'^\mu + \epsilon^\mu(x')$ into the definition of the transformation matrix $P^\mu_{\nu'}$

$$P^\alpha_{\mu'} = \frac{\partial x^\alpha}{\partial x'^{\mu'}} = \frac{\partial x'^\alpha + \epsilon^\alpha(x')}{\partial x'^{\mu'}} = \delta^\alpha_{\mu'} + \partial_{\mu'} \epsilon^\alpha(x'). \quad (43)$$

For the metric $g_{\alpha\beta}(x)$, we do a Taylor expansion,

$$\begin{aligned}g_{\alpha\beta}(x) &= g_{\alpha\beta}(x' + \epsilon(x')) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{\partial^n g_{\alpha\beta}(x' + \epsilon(x'))}{\partial^n (x'^\gamma + \epsilon^\gamma(x'))} \right|_{x'^\gamma + \epsilon^\gamma(x') = x'^\gamma} (x'^\gamma + \epsilon^\gamma(x') - x'^\gamma)^n \\ &= g_{\alpha\beta}(x') + \epsilon^\gamma(x') \partial_{\gamma'} g_{\alpha\beta}(x') + \mathcal{O}[\epsilon^2(x')].\end{aligned}$$

Inserting eq. (43) and part ii) into eq. (42), we get

$$\begin{aligned}g_{\mu'\nu'}(x') &= [\delta^\alpha_{\mu'} + \partial_{\mu'} \epsilon^\alpha(x')][\delta^\beta_{\nu'} + \partial_{\nu'} \epsilon^\beta(x')][g_{\alpha\beta}(x') + \epsilon^\gamma(x') \partial_{\gamma'} g_{\alpha\beta}(x') + \mathcal{O}[\epsilon^2(x')]] \\ &= \delta^\alpha_{\mu'} \delta^\beta_{\nu'} g_{\alpha\beta}(x') + [\partial_{\mu'} \epsilon^\alpha(x')][\delta^\beta_{\nu'} g_{\alpha\beta}(x') + \delta^\alpha_{\mu'} [\partial_{\nu'} \epsilon^\beta(x')] g_{\alpha\beta}(x')] \\ &\quad + \delta^\alpha_{\mu'} \delta^\beta_{\nu'} \epsilon^\gamma(x') \partial_{\gamma'} g_{\alpha\beta}(x') + \mathcal{O}[\epsilon^2(x')] \\ &= g_{\mu\nu}(x') + g_{\alpha\nu}(x') \partial_{\mu'} \epsilon^\alpha(x') + g_{\mu\beta}(x') \partial_{\nu'} \epsilon^\beta(x') + \epsilon^\gamma(x') \partial_{\gamma'} g_{\mu\nu}(x') + \mathcal{O}[\epsilon^2(x')].\end{aligned}$$

From this we infer that

$$g_{\mu'\nu'}(x) = g_{\mu\nu}(x) + g_{\alpha\nu}(x) \partial_\mu \epsilon^\alpha(x) + g_{\mu\beta}(x) \partial_\nu \epsilon^\beta(x) + \epsilon^\gamma(x) \partial_\gamma g_{\mu\nu}(x) + \mathcal{O}[\epsilon^2(x)], \quad (44)$$

and hence

$$\begin{aligned}\delta g_{\mu\nu}(x) &= g_{\mu'\nu'}(x) - g_{\mu\nu}(x) \\ &= g_{\alpha\nu}(x) \partial_\mu \epsilon^\alpha(x) + g_{\mu\beta}(x) \partial_\nu \epsilon^\beta(x) + \epsilon^\gamma(x) \partial_\gamma g_{\mu\nu}(x) + \mathcal{O}[\epsilon^2(x)].\end{aligned}\quad (45)$$

To see that the second equality in ii) holds, we simply calculate

$$\begin{aligned}
\nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu &= \nabla_\mu (g_{\alpha\nu} \epsilon^\alpha) + \nabla_\nu (g_{\mu\beta} \epsilon^\beta) = g_{\alpha\nu} \nabla_\mu \epsilon^\alpha + g_{\mu\beta} \nabla_\nu \epsilon^\beta + \underbrace{\epsilon^\gamma \nabla_\gamma g_{\mu\nu}}_0 \\
&= g_{\alpha\nu} (\partial_\mu \epsilon^\alpha + \Gamma^\alpha_{\mu\delta} \epsilon^\delta) + g_{\mu\beta} (\partial_\nu \epsilon^\beta + \Gamma^\beta_{\nu\zeta} \epsilon^\zeta) \\
&\quad + \epsilon^\gamma (\partial_\gamma g_{\mu\nu} - \Gamma^\alpha_{\mu\gamma} g_{\alpha\nu} - \Gamma^\beta_{\nu\gamma} g_{\mu\beta}) \\
&= g_{\alpha\nu} \partial_\mu \epsilon^\alpha + g_{\mu\beta} \partial_\nu \epsilon^\beta + \epsilon^\gamma \partial_\gamma g_{\mu\nu} \\
&\quad + g_{\alpha\nu} \Gamma^\alpha_{\mu\delta} \epsilon^\delta - \epsilon^\gamma \Gamma^\alpha_{\mu\gamma} g_{\alpha\nu} + g_{\mu\beta} \Gamma^\beta_{\nu\zeta} \epsilon^\zeta - \epsilon^\gamma \Gamma^\beta_{\nu\gamma} g_{\mu\beta} \\
&= g_{\alpha\nu} \partial_\mu \epsilon^\alpha + g_{\mu\beta} \partial_\nu \epsilon^\beta + \epsilon^\gamma \partial_\gamma g_{\mu\nu} = \delta g_{\mu\nu} - \mathcal{O}[\epsilon^2].
\end{aligned} \tag{46}$$

iii) By part a), we have

$$\sqrt{-\det[\mathbf{g}(x)]} \rightarrow \sqrt{-\det[\mathbf{g}'(x')]} = J_{x \rightarrow x'} \sqrt{-\det[\mathbf{g}(x)]} = J_{x \rightarrow x'} g(x), \tag{47}$$

where to save on writing, we introduced the shorthand notation $\sqrt{-\det[\mathbf{g}(x)]} = g(x)$. To find the infinitesimal variation $\delta \sqrt{-\det[\mathbf{g}(x)]} = \delta g(x)$, we expand eq. (47) to first order in $\epsilon(x)$,

$$\begin{aligned}
g'(x') &= J_{x \rightarrow x'} g(x) = \det(\mathbf{P}_{x \rightarrow x'}) g(x' + \epsilon(x')) \\
&= \underbrace{\det[\mathbf{1} + \boldsymbol{\partial}_{x'} \boldsymbol{\epsilon}(x')]}_{1 + \text{Tr}[\boldsymbol{\partial}_{x'} \boldsymbol{\epsilon}(x')] + \mathcal{O}[\epsilon^2(x')]} \left(g(x') + \epsilon^\gamma(x') \partial_{\gamma'} g(x') + \mathcal{O}[\epsilon^2(x')] \right) \\
&= g(x') + [\partial_{\mu'} \epsilon^\mu(x')] g(x') + \epsilon^\mu(x') \partial_{\mu'} g(x') + \mathcal{O}[\epsilon^2(x')] \\
&= g(x') + \partial_{\mu'} [\epsilon^\mu(x') g(x')] + \mathcal{O}[\epsilon^2(x')],
\end{aligned} \tag{48}$$

where to get to the second line we used $\mathbf{P}_{x \rightarrow x'} = \mathbf{1} + \boldsymbol{\partial}_{x'} \boldsymbol{\epsilon}(x')$, i.e. eq. (43) in matrix form. From eq. (48), we infer

$$g'(x) = g(x) + \partial_\mu [\epsilon^\mu(x) g(x)] + \mathcal{O}[\epsilon^2(x)], \tag{49}$$

from which in turn it follows that

$$\delta g(x) = g'(x) - g(x) = \partial_\mu [\epsilon^\mu(x) g(x)] + \mathcal{O}[\epsilon^2(x)], \tag{50}$$

or in unshortened notation,

$$\delta \sqrt{-\det[\mathbf{g}(x)]} = \partial_\mu [\epsilon^\mu(x) \sqrt{-\det[\mathbf{g}(x)]}] + \mathcal{O}[\epsilon^2(x)]. \tag{51}$$

3 Action Principle

a) Consider a field $\varphi(x)$ and an action in d spacetime dimensions with Minkowskian signature $(-1, +1, \dots, +1)$ of the form

$$S[\varphi] = \int d^d x \mathcal{L}(\varphi(x), \partial_\mu \varphi(x)). \tag{52}$$

The simplest example is the action for the free scalar field

$$S[\varphi] = -\frac{1}{2} \int d^d x \left(\partial_\mu \varphi \partial^\mu \varphi + m^2 \varphi^2 \right). \tag{53}$$

The variation of the general action (52) is defined as

$$\delta S[\varphi] = \int d^d x \left(\frac{\partial \mathcal{L}}{\partial \varphi} \delta \varphi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \delta (\partial_\mu \varphi) \right). \tag{54}$$

Using integration by parts and neglecting boundary terms show that the action principle $\delta S[\varphi] = 0$ implies the Euler Lagrange equations

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = \frac{\partial \mathcal{L}}{\partial \varphi}. \quad (55)$$

- b) Use this result to derive the Klein-Gordon equation as the Euler-Lagrange equation of the free scalar field action (53).

- a) Using partial integration on the second term in the variation of the action (54), $\delta S[\varphi]$ becomes

$$\delta S[\varphi] = \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta \varphi \Big|_{\partial \mathbb{R}^d}}_0 + \int d^d x \left(\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \delta \varphi. \quad (56)$$

Since we assume we are working with localized *physical* systems, we take the boundary terms to vanish at spatial and temporal infinity, i.e. the term in front can be disregarded.

At this point, we resort to Hamilton's principle of a stationary action, $\delta S[\varphi] = 0$, which we require to hold for all possible variations $\delta \varphi$ of $\varphi(x)$. $\delta S[\varphi] = 0$ can hence only be true in general if the integrand itself vanishes. We thus arrive at the renowned Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = 0. \quad (57)$$

- b) Comparing eqs. (52) and (53), we see that the Lagrangian $\mathcal{L}(\varphi(x), \partial_\mu \varphi(x))$ of the free scalar field is given by

$$\mathcal{L}(\varphi(x), \partial_\mu \varphi(x)) = -\frac{1}{2} \left(\partial_\mu \varphi \partial^\mu \varphi + m^2 \varphi^2 \right). \quad (58)$$

Inserting this expression into eq. (57), we obtain as equation of motion,

$$-m^2 \varphi + \partial_\mu \partial^\mu \varphi = 0, \quad (59)$$

which written in a more familiar form reads

$$(\Box_x - m^2) \varphi(x) = 0. \quad (60)$$