

Quantum Field Theory II - Assignment 2Problem 2.1 (Path integral and time-ordering)

Show, by using the steps of the derivation of the path integral presented in the lecture, that

$$\int_{q(t_1)=q_i}^{q(t_2)=q_f} Dq(t) Dp(t) q(t_1) q(t_2) e^{iS(p,q)} = \langle q_f, t_f | T q(t_1) q(t_2) | q_i, t_i \rangle. \quad (1)$$

Put special emphasis on showing how the time-ordering appears.

$$\begin{aligned} \int_{q(t_1)=q_i}^{q(t_2)=q_f} Dq(t) Dp(t) q(t_1) q(t_2) e^{iS(p,q)} &= \int_{q(t_1)=q_i}^{q(t_2)=q_f} Dq(t) Dp(t) q(t_1) q(t_2) e^{i \int_{t_1}^{t_2} dt L(p,q)} \\ &= \int_{q(t_1)=q_i}^{q(t_2)=q_f} Dq(t) Dp(t) q(t_1) q(t_2) e^{i \int_{t_1}^{t_2} dt (p \dot{q} - H(p,q))} = \lim_{N \rightarrow \infty} \int \frac{dp_1}{2\pi} \prod_{k=1}^N \frac{dp_k dq_k}{2\pi} q(t_1) q(t_2) e^{i \sum_{k=0}^N p_k q_{k+1}} e^{-i \sum_{k=0}^N H(p_k, \frac{q_k + q_{k+1}}{2}) \delta t} \end{aligned}$$

where $q_0 \equiv q_i$ and $q_{N+1} \equiv q_f$. The derivative of q became $\dot{q} \rightarrow \frac{q_{k+1} - q_k}{\delta t}$ for $\delta t > 0$ discrete

$$= \lim_{N \rightarrow \infty} \int \frac{dp_1}{2\pi} \prod_{k=1}^N \frac{dp_k dq_k}{2\pi} q(t_1) q(t_2) e^{i \sum_{k=0}^N p_k q_{k+1}} e^{-i \sum_{k=0}^N H(p_k, \frac{q_k + q_{k+1}}{2}) \delta t}$$

We inserted $\bar{q}_k = \frac{1}{2}(q_k + q_{k+1})$ when discretizing the time t and now require $H(p,q)$ to decompose such that $H(p,q) = f(p) + V(q)$.

$$= \lim_{N \rightarrow \infty} \int dp_1 \prod_{k=1}^N dp_k dq_k q(t_1) q(t_2) \prod_{k=0}^N \langle q_{k+1} | p_k \rangle \langle p_k | q_k \rangle e^{i \sum_{k=0}^N f(p_k) \delta t} e^{-i \sum_{k=0}^N V(q_k) \delta t}$$

where we used $\frac{1}{\sqrt{2\pi}} e^{ip_k q_{k+1}} = \langle q_{k+1} | p_k \rangle$ and $\frac{1}{\sqrt{2\pi}} e^{-ip_k q_k} = \langle p_k | q_k \rangle$.

$$\begin{aligned} &= \lim_{N \rightarrow \infty} \int dp_1 \prod_{k=1}^N dp_k dq_k q(t_1) q(t_2) \prod_{k=0}^N \langle q_{k+1} | e^{i f(p_k) \delta t} | p_k \rangle \langle p_k | e^{-i V(q_k) \delta t} | q_k \rangle \\ &= \lim_{N \rightarrow \infty} \int dp_1 \prod_{k=1}^N dp_k dq_k q(t_1) q(t_2) \prod_{k=0}^N \langle q_{k+1} | e^{-i f(\bar{p}) \delta t} | p_k \rangle \langle p_k | e^{-i V(\bar{q}) \delta t} | q_k \rangle \\ &= \lim_{N \rightarrow \infty} \int \prod_{k=1}^N dq_k q(t_1) q(t_2) \prod_{k=0}^N \langle q_{k+1} | e^{-i f(\bar{p}) \delta t} e^{-i V(\bar{q}) \delta t} | q_k \rangle. \end{aligned}$$

Here, we insert a zero in the form of $e^{\frac{i}{2} f(\bar{p}) \delta t + i V(\bar{q}) \delta t + o(\delta t^2)}$, which vanishes when performing the limit $\lim_{N \rightarrow \infty}$. We can apply the Baker-Campbell-Hausdorff formula backwards to get

We now move on to a shorter and more elegant proof of eq. (4) that is also more intrinsic to the path integral formulation. It works via dissection of the path integral. We operate here under the assumption that $t_2 > t_1 > t_3$ or $t_1 > t_2 > t_3 > t_4$, i.e. that t_1 is truly larger/smaller than t_2 . If $t_1 = t_2$, time-ordering is obsolete and there is nothing to show.

$$\int_{q(t_1)=q_F}^{q(t_2)=q_F} Dq(t) Dp(t) q(t_1) q(t_2) e^{iS(p,q)} := \lim_{N \rightarrow \infty} \int_{\frac{q_F}{2N}}^{\frac{q_F}{2N}} \prod_{k=1}^N \frac{dq_k}{2\pi} q(t_1) q(t_2) e^{i \sum_{k=0}^N (p_k \frac{q_{k+1} - q_k}{\delta t} - H(p_k, \bar{q}_k) \delta t)} = \int_{\frac{q_F}{2N}}^{\frac{q_F}{2N}} Dq_k \prod_{k=1}^N \frac{dq_k}{2\pi} q(t_2) e^{i \sum_{k=0}^N L(p_k, \bar{q}_k)}$$

The product $\prod_{k=1}^N$ we now split up at j such that $j \delta t = \frac{1}{2}(t_1 - t_2)$. This ensures that $q(t_1)$ and $q(t_2)$ can each be evaluated solely in one of the two resulting integrals and that they can be evaluated separately from one another. To actually perform the path integral split up, we may trouble Fubini's theorem and refer to the well-definedness of the path integral in quantum mechanics as demonstrated in section 2.1 of the script. To save on writing, we hence assume $t_1 > t_2$.

$$= \lim_{N \rightarrow \infty} \int_{\frac{q_F}{2N}}^{\frac{q_F}{2N}} \prod_{k=1}^j \frac{dq_k}{2\pi} q(t_1) e^{i \sum_{k=0}^j L(p_k, \bar{q}_k)} \int_{\frac{q_F}{2N}}^{\frac{q_F}{2N}} \prod_{k=j+1}^N \frac{dq_k}{2\pi} q(t_2) e^{i \sum_{k=j+1}^N L(p_k, \bar{q}_k)}$$

In the above line, we note that our split up created two new path integrals and an integral over q_{j+1} in between. The expression can therefore be rewritten, again using Fubini's theorem to rearrange integrals, as

$$\begin{aligned} &= \int_{q(t_1)=q_F}^{q(t_1)=q_F} Dq(t) Dp(t) q(t_1) e^{iS(p,q)} \int_{q(t_2)=q_F}^{q(t_2)=q_F} Dq(t) Dp(t) q(t_2) e^{iS(p,q)} \\ &= \int dq_j \langle q_{F,t_1} | \hat{q}_N(t_1) | q_{j,t_1} \rangle \langle q_{j,t_2} | \hat{q}_N(t_2) | q_{F,t_2} \rangle \\ &= \int dq_j \langle q_{F,t_1} | \hat{q}_N(t_1) e^{iH(t_1)} | q_j \rangle \langle q_j | e^{-iH(t_2)} \hat{q}_N(t_2) | q_{F,t_2} \rangle = \langle q_{F,t_1} | \hat{q}_N(t_1) e^{iH(t_1)} e^{-iH(t_2)} \hat{q}_N(t_2) | q_{F,t_2} \rangle \\ &= \langle q_{F,t_1} | \hat{q}_N(t_1) \hat{q}_N(t_2) | q_{F,t_2} \rangle = \langle q_{F,t_1} | T \hat{q}_N(t_1) \hat{q}_N(t_2) | q_{F,t_2} \rangle, \end{aligned}$$

where in the last step, time-ordering could be added trivially since we set $t_1 > t_2$. However, it is important to note that we could equally well have set

$t_2 > t_1$, in which case following the same procedure as we did just now, $q(t_2)$ would have ended up in the left path integral and $q(t_1)$ in the right one, reversing their order in the final expression. This goes to show that the path integral formulation of quantum field theory takes care of time-ordering automatically.

Problem 2.2 (Quantum mechanical oscillator and path integral)

The transition amplitude between a state $|q_a\rangle$ at $t=0$ and $|q_b\rangle$ at $t=T$ can be schematically expressed as

$$\langle q_b | e^{-iHt} | q_a \rangle = \int_{q(0)=q_a}^{q(T)=q_b} Dq e^{iS[q]} \quad (2)$$

where the integral $\int Dq$ is performed over all the possible trajectories connecting the points $q(0) = q_a$ and $q(T) = q_b$, and $S[q] = \int_0^T dt \mathcal{L}(q, \dot{q})$. In the case of free fields, $H = \frac{p^2}{2m}$, the transition amplitude is simple to calculate. Show that

$$\langle q_b | e^{-i\frac{p^2}{2m}T} | q_a \rangle = \sqrt{\frac{m}{2\pi i T}} e^{i\frac{m}{2T}(q_b - q_a)^2}$$

$$\begin{aligned} \langle q_b | e^{-i\frac{p^2}{2m}T} | q_a \rangle &= \langle q_b | e^{-i\frac{p^2}{2m}T} \int dp |p\rangle \langle p| q_a \rangle = \int dp \langle q_b | e^{-i\frac{p^2}{2m}T} | p \rangle \langle p | q_a \rangle \\ &\quad \text{by the completeness relation} \\ &= \int dp \langle q_b | e^{-i\frac{p^2}{2m}T} | p \rangle \langle p | q_a \rangle = \int dp \underbrace{\langle q_b | p \rangle}_{\frac{1}{\sqrt{2\pi}} e^{ipq_b}} \underbrace{\langle p | q_a \rangle}_{\frac{1}{\sqrt{2\pi}} e^{-ipq_a}} e^{-i\frac{p^2}{2m}T} = \int \frac{dp}{2\pi} e^{-ip(q_b - q_a)} e^{-i\frac{p^2}{2m}T} \\ &= \int \frac{dp}{2\pi} \exp\left[-\frac{iT}{2m}\left(p + \frac{m}{T}(q_b - q_a)\right)^2 + i\frac{m}{2T}(q_b - q_a)^2\right], \quad \text{Gauss integral: } \int_{-\infty}^{\infty} e^{-a(x+b)^2} dx = \sqrt{\frac{\pi}{a}} \\ &= \frac{1}{2\pi} \sqrt{\frac{\pi}{\frac{iT}{2m}}} e^{i\frac{m}{2T}(q_b - q_a)^2} = \sqrt{\frac{m}{2\pi i T}} e^{i\frac{m}{2T}(q_b - q_a)^2} \end{aligned}$$

We are now going to determine in a few steps the transition amplitude of an harmonic oscillator. The system is characterized by the Lagrangian density

$$\mathcal{L} = \frac{m}{2} \dot{q}^2 - \frac{m\omega^2}{2} q^2$$

a) Recall that the classical trajectory $q(t)$ is found by extremising the action,

$$\frac{\delta S[q(t)]}{\delta q(t)} = -m\ddot{q}(t) - m\omega^2 q(t) = 0,$$

and imposing the boundary conditions $q(0) = q_a$ and $q(T) = q_b$. An arbitrary

trajectory $q(t)$ can then be decomposed as $q(t) = q_c(t) + y(t)$ with the boundary conditions $y(0) = y(T) = 0$.

$$\begin{aligned} \frac{\delta S[q(t)]}{\delta q(t)} &= \frac{\delta}{\delta q(t)} \int_0^T dt' \mathcal{L} = \frac{\delta}{\delta q(t)} \int_0^T dt' \left(\frac{m}{2} \dot{q}^2(q, t') - \frac{m}{2} \omega^2 q^2(t') \right) = \int_0^T dt' \left(\frac{m}{2} \frac{\delta \dot{q}^2(q, t')}{\delta q(t)} - \frac{m}{2} \omega^2 \frac{\delta q^2(t')}{\delta q(t)} \right) \\ &= \int_0^T dt' \left(m \dot{q}(q, t') \frac{\delta \dot{q}(t')}{\delta q(t)} - m \omega^2 q(t') \frac{\delta q(t')}{\delta q(t)} \right) = \int_0^T dt' \left(m \dot{q}(q, t') \frac{d}{dt'} \frac{\delta q(t')}{\delta q(t)} - m \omega^2 q(t') \delta(t-t') \right) \\ &= \underbrace{m \dot{q}(q, t') \frac{\delta q(t')}{\delta q(t)}}_0 \Big|_0^T + \int_0^T dt' \left(m \frac{d}{dt'} \dot{q}(q, t') - m \omega^2 q(t') \right) \delta(t-t') \\ &= -m \ddot{q}(q, t) - m \omega^2 q(t) = 0 \implies m \ddot{q} + m \omega^2 q = 0 \end{aligned}$$

The boundary terms coming from the partial integration vanish, because the boundary points $q(t=0) = q_0$ and $q(t=T) = q_1$ are held fixed, meaning their variation vanishes, i.e.

$$\frac{\delta q(t=T)}{\delta q(t)} = \frac{\delta q(t=0)}{\delta q(t)} = 0.$$

b) Convince yourself that the following expression for the action is exact

$$S[q] = S[q_c] + \int_0^T dt \frac{\delta S[q]}{\delta q(t)} \Big|_{q(t)=q_c(t)} y(t) + \frac{1}{2} \int_0^T dt \int_0^T dt' \frac{\delta^2 S[q]}{\delta q(t) \delta q(t')} \Big|_{q(t)=q_c(t)} y(t) y(t'). \quad (4)$$

Then show that we can write

$$\frac{\delta^2 S[q]}{\delta q(t) \delta q(t')} = -m \frac{d^2}{dt^2} \delta(t-t') - m \omega^2 \delta(t-t') \quad (5)$$

so that

$$S[q] = S[q_c] + \frac{m}{2} \int_0^T dt (y''(t) - \omega^2 y(t)) = S[q_c] + S[y]. \quad (6)$$

Then our initial amplitude reads

$$\int Dq e^{iS[q]} = e^{iS[q_c]} \int Dy e^{iS[y]}. \quad (7)$$

To do the convincing, we choose to first show eq. (5). In part a), we

already showed, that $\frac{\delta S[q]}{\delta q(t)} = -m\ddot{q}(t) - m\omega^2 q(t)$. Therefore

$$\begin{aligned}\frac{\delta^2 S[q]}{\delta q(t) \delta q(t')} &= \frac{\delta}{\delta q(t)} \frac{\delta S[q]}{\delta q(t')} = \frac{\delta}{\delta q(t)} (-m\ddot{q}(t') - m\omega^2 q(t')) = -m \frac{\delta \ddot{q}(t')}{\delta q(t)} - m\omega^2 \frac{\delta q(t')}{\delta q(t)} \\ &= -m \frac{d^2}{dt^2} \frac{\delta q(t')}{\delta q(t)} - m\omega^2 \frac{\delta q(t')}{\delta q(t)} = -m \frac{d^2}{dt^2} \delta(t-t') - m\omega^2 \delta(t-t')\end{aligned}$$

Using this result, we can make sure eq. (4) is correct

$$\begin{aligned}S[q] &= \int_0^T dt \mathcal{L}(q(t), \dot{q}(t), t) = \int_0^T dt \left(\frac{m}{2} (\dot{q}_c(t) + \dot{y}(t))^2 - \frac{m}{2} \omega^2 (q_c(t) + y(t))^2 \right) \\ &= \underbrace{\int_0^T dt \left(\frac{m}{2} \dot{q}_c^2(t) - \frac{m}{2} \omega^2 q_c^2(t) \right)}_{S[q_c]} + \int_0^T dt \left(m \dot{q}_c(t) \dot{y}(t) - m \omega^2 q_c(t) y(t) \right) + \int_0^T dt \left(\frac{m}{2} \dot{y}^2(t) - \frac{m}{2} \omega^2 y^2(t) \right) \\ &= S[q_c] + \underbrace{m \dot{q}_c(t) y(t) \Big|_0^T}_{0, \text{ since } y(0)=y(T)=0} + \int_0^T dt \underbrace{\left(-m \ddot{q}_c(t) - m \omega^2 q_c(t) \right)}_{\frac{\delta S[q]}{\delta q(t)}} y(t) \\ &\quad + \frac{1}{2} \int_0^T dt \int_0^T dt' \left(m \dot{y}(t) \dot{y}(t') \delta(t-t') - m \omega^2 y(t) y(t') \delta(t-t') \right) \\ &= S[q_c] + \int_0^T dt \frac{\delta S[q]}{\delta q(t)} y(t) + \frac{1}{2} m y(t) \int_0^T dt' \dot{y}(t') \delta(t-t') + \frac{1}{2} \int_0^T dt \int_0^T dt' \left(-m y(t) \dot{y}(t') \frac{d}{dt'} \delta(t-t') - \right. \\ &\quad \left. - m \omega^2 y(t) y(t') \delta(t-t') \right) \\ &= S[q_c] + \int_0^T dt \frac{\delta S[q]}{\delta q(t)} y(t) - \frac{1}{2} m y(t) \int_0^T dt y(t) \frac{d}{dt} \delta(t-t') \Big|_0^T \\ &\quad + \frac{1}{2} \int_0^T dt \int_0^T dt' \left(m y(t) y(t') \frac{d^2}{dt^2} \delta(t-t') - m \omega^2 y(t) y(t') \delta(t-t') \right) \\ &= S[q_c] + \int_0^T dt \frac{\delta S[q]}{\delta q(t)} y(t) + \frac{1}{2} \int_0^T dt \int_0^T dt' \frac{\delta^2 S[q]}{\delta q(t) \delta q(t')} y(t) y(t')\end{aligned}$$

A simpler and more general way of proving the exactness of eq. (4) is acknowledging that it marks the Taylor expansion of $S[q]$ w.r.t. $y(t)$ at $q_c(t)$, the classical trajectory, where all terms above the quadratic one vanish due to our particular form of Lagrangian, which contains only quadratic terms in $q(t)$ and $\dot{q}(t)$, such that

$$\frac{\delta^3 S[q]}{\delta q(t) \delta q(t') \delta q(t'')} = \frac{\delta}{\delta q(t)} \left(-m \frac{d^2}{dt^2} \delta(t-t'') - m \omega^2 \delta(t-t'') \right) = 0.$$

In any case, given eq. (4), it is easy to deduce eq. (6). Since the term linear in $y(t)$ vanishes due to $\frac{\delta S(q)}{\delta y(t)} = 0$, we get

$$S[q] = S[q_c] + \frac{1}{2} \int_0^T dt' \left(-m \frac{d}{dt'} \delta(t-t') - m\omega^2 \delta(t-t') \right) y(t) y(t')$$
$$= S[q_c] - \frac{1}{2} m \int_0^T dt' \left(\frac{d}{dt'} \delta(t-t') \right) y(t) y(t') = \frac{1}{2} \int_0^T dt' \left(m \frac{d}{dt'} \delta(t-t') \dot{y}(t) y(t') - m\omega^2 \delta(t-t') y(t) y(t') \right)$$
$$\quad \quad \quad 0, \text{ since } y(0) = y(T) = 0$$
$$= S[q_c] + \frac{1}{2} \int_0^T dt \int_0^T dt' \left(-m \delta(t-t') \ddot{y}(t) y(t') - m\omega^2 \delta(t-t') y(t) y(t') \right)$$
$$= S[q_c] - \frac{m}{2} \int_0^T dt \left(\ddot{y}(t) + \omega^2 y(t) \right) y(t) = S[q_c] - \underbrace{\frac{m}{2} \int_0^T dt (\dot{y}^2(t))}_{} + \underbrace{\frac{m}{2} \int_0^T dt (-\omega^2 y^2(t))}_{= S[y]}$$
$$\quad \quad \quad (= S[y]) \quad \quad \quad (= S[y]) \quad \quad \quad (= S[y])$$

From this result, we see that $\int Dq e^{iS[q]} = e^{iS[q_0]} \int Dy e^{iS[y]}$

c) Show that

$$S(q, \omega) = \frac{m\omega}{2\sin(\omega T)} [(d_1^2 + q_2^2) \cos(\omega T) - 2q_1 q_2] \quad (8)$$

Hint: T is not connected to w by a relation of the sort $T \propto \frac{1}{2\pi w}$

The classical trajectory, $q_c(t)$, is the one that extremizes the action

$S[d]$. As shown in eq. (2), for $\frac{S[d]}{S[d] + q[d]} = 0$, it holds that

$$\ddot{q}_c(t) = -\omega^2 q_c(t),$$

of which the general solution is $q(t) = A \sin(\omega(t-t_0)) + B \cos(\omega(t-t_0))$. Since the harmonic oscillator is time-translation invariant, we may set $t_0 = 0$ and use the boundary conditions for $q(0)$ and $q(T)$ to determine A and B .

$$q_n := q(t=0) = A \sin(0) + B \cos(0) = B,$$

$$q_1 := q(t=T) = A \sin(\omega T) + q_0 \cos(\omega T) \Rightarrow A = \frac{q_0 - q_0 \cos(\omega T)}{\sin(\omega T)}$$

For the case of our harmonic oscillator, the classical path $q_c(t)$ is hence given by

$$q_c(t) = \frac{q_0 - q_0 \cos(\omega t)}{\sin(\omega t)} \sin(\omega t) + q_0 \cos(\omega t)$$

Inserting the classical path into the Lagrangian and integrating from 0 to T yields

$$\begin{aligned}
 S[q_c] &= \int_0^T dt \mathcal{L}|_{q(t)=q_c(t)} = \int_0^T dt \left(\frac{m}{2} \dot{q}_c^2(t) - \frac{m}{2} \omega^2 q_c^2(t) \right) \\
 &= \int_0^T dt \left(\frac{m}{2} \omega^2 \left(A \cos(\omega t) - B \sin(\omega t) \right)^2 - \frac{m}{2} \omega^2 \left(A \sin(\omega t) + B \cos(\omega t) \right)^2 \right) \\
 &= \frac{m}{2} \omega^2 \int_0^T dt \left(A^2 \cos^2(\omega t) - 2AB \sin(\omega t) \cos(\omega t) + B^2 \sin^2(\omega t) - A^2 \sin^2(\omega t) \right. \\
 &\quad \left. - 2AB \sin(\omega t) \cos(\omega t) - B^2 \cos^2(\omega t) \right) \\
 &= \frac{m}{2} \omega^2 \int_0^T dt \left(A^2 (\cos^2(\omega t) - \sin^2(\omega t)) - B^2 (\cos^2(\omega t) - \sin^2(\omega t)) - 2AB \sin(2\omega t) \right),
 \end{aligned}$$

where we used that $\sin(\alpha) \cos(\beta) \pm \sin(\beta) \cos(\alpha) = \sin(\alpha \pm \beta)$. In the next step, we'll employ $\cos(\alpha) \cos(\beta) \mp \sin(\alpha) \sin(\beta) = \cos(\alpha \pm \beta)$ to get

$$\begin{aligned}
 S[q_c] &= \frac{m}{2} \omega^2 \int_0^T dt \left((A^2 - B^2) \cos(2\omega t) - 2AB \sin(2\omega t) \right) \\
 &= \frac{m}{2} \omega^2 \left(\frac{A^2 - B^2}{2\omega} \sin(2\omega t) \Big|_0^T + \frac{2AB}{2\omega} \cos(2\omega t) \Big|_0^T \right) \\
 &= \frac{m}{4} \omega \left((A^2 - B^2) \sin(2\omega T) + 2AB \frac{\cos(2\omega T) - 1}{2\sin(\omega T)} \right),
 \end{aligned}$$

where $A^2 - B^2 = \frac{q_0^2 - 2q_0 q_T \cos(\omega T) + q_T^2 (\cos^2(\omega T) - \sin^2(\omega T))}{\sin^2(\omega T)} - q_0^2$, $AB = \frac{q_0 q_T - q_0^2 \cos(\omega T)}{\sin(\omega T)}$

$$\begin{aligned}
 S[q_c] &= \frac{m\omega}{2\sin(\omega T)} \left(\frac{q_0^2 - 2q_0 q_T \cos(\omega T) + q_T^2 (\cos^2(\omega T) - \sin^2(\omega T))}{2\sin(\omega T) \sin^2(\omega T)} + 2 \frac{(q_0 q_T - q_0^2 \cos(\omega T)) \sin(\omega T)}{\sin^2(\omega T)} \right) \\
 &= \frac{m\omega}{2\sin(\omega T)} \left(\frac{1}{2\sin(\omega T)} \left((q_0^2 - 2q_0 q_T \cos(\omega T) + q_T^2 (\cos^2(\omega T) - \sin^2(\omega T))) + 4(q_0 q_T - q_0^2 \cos(\omega T)) \sin^2(\omega T) \right) \right) \\
 &= \frac{m\omega}{2\sin(\omega T)} \left((q_0^2 + q_T^2) \cos(\omega T) - 2q_0 q_T \right)
 \end{aligned}$$

d) At this point, it is convenient to introduce functions

$$y_n(t) = C_n \sin\left(\frac{n\pi t}{T}\right), \quad (9)$$

where C_n are constants to be determined. The $y_n(t)$ are orthonormal on the interval $[0, T]$:

$$\int_0^T dt y_n(t) y_m(t) = \delta_{nm}.$$

These can be used as a basis to expand any function $y(t)$, satisfying our boundary conditions, i.e.

$$y(t) = \sum_{n=1}^{\infty} a_n y_n(t), \quad (10)$$

by means of a set of constants a_n . Then show that

$$S[y] = \frac{m}{2} \sum_{n=1}^{\infty} \lambda_n a_n^2 \quad (11)$$

and determine the constant quantities λ_n .

The ζ_n are constrained by the required orthogonality.

$$\begin{aligned} \int_0^T dt y_n(t) y_m(t) &= \int_0^T dt C_n \sin\left(\frac{n\pi t}{T}\right) C_m \sin\left(\frac{m\pi t}{T}\right) = \int_0^T dt C_n C_m \cos\left(\frac{1}{2}\left((n-m)\frac{\pi t}{T} - (n+m)\frac{\pi t}{T}\right)\right) \\ &= \frac{1}{2} C_n C_m \left(\frac{T}{n-m} \sin\left((n-m)\frac{\pi}{T}\right) - \frac{T}{n+m} \sin\left((n+m)\frac{\pi}{T}\right) \right) \Big|_0^T \\ &= \frac{T}{2\pi} C_n C_m \left(\frac{\sin((n-m)\pi)}{n-m} - \frac{\sin((n+m)\pi)}{n+m} \right) = \frac{T}{2\pi} C_n C_m \frac{\sin((n-m)\pi)}{n-m} \stackrel{!}{=} \delta_{nm} = \begin{cases} 1 & \text{if } n=m \\ 0 & \text{else} \end{cases} \end{aligned}$$

Since $\lim_{n \rightarrow m} \frac{\sin((n-m)\pi)}{n-m} = 1$, C_n needs to fulfill $\frac{T}{2\pi} C_n^2 = 1$ or $C_n = \sqrt{\frac{2\pi}{T}}$. Therefore, our expansions of $y(t)$ and $\dot{y}(t)$ read

$$y(t) = \sum_{n=1}^{\infty} a_n y_n(t) = \sum_{n=1}^{\infty} a_n C_n \sin\left(\frac{n\pi t}{T}\right) = \sum_{n=1}^{\infty} a_n \sqrt{\frac{2\pi}{T}} \sin\left(\frac{n\pi t}{T}\right)$$

$$\dot{y}(t) = \sum_{n=1}^{\infty} a_n \dot{y}_n(t) = \sum_{n=1}^{\infty} a_n C_n \frac{n\pi}{T} \sin\left(\frac{n\pi t}{T}\right) = \sum_{n=1}^{\infty} a_n \sqrt{\frac{2\pi}{T}} \frac{n\pi}{T} \sin\left(\frac{n\pi t}{T}\right) = - \sum_{n=1}^{\infty} a_n \frac{n^2 \pi^2}{T^2} y_n(t)$$

Inserting this into the action of quantum fluctuations, $S[y]$, we get

$$\begin{aligned} S[y] &= \frac{m}{2} \int_0^T dt (\dot{y}^2(t) - \omega^2 y^2(t)) = - \int_0^T dt (\ddot{y}(t) + \omega^2 y(t)) y(t) \\ &= - \frac{m}{2} \int_0^T dt \sum_{n,m=1}^{\infty} \left(-a_n \frac{n^2 \pi^2}{T^2} y_n(t) + \omega^2 a_n y_n(t) \right) a_m y_m(t) \\ &= - \frac{m}{2} \sum_{n,m=1}^{\infty} \left(-a_n a_m \frac{n^2 \pi^2}{T^2} \delta_{nm} + a_n a_m \omega^2 \delta_{nm} \right) = \frac{m}{2} \sum_{n=1}^{\infty} \left(a_n^2 \frac{n^2 \pi^2}{T^2} + a_n^2 \omega^2 \right) \\ &= \frac{m}{2} \sum_{n=1}^{\infty} \underbrace{\left(\frac{n^2 \pi^2}{T^2} + \omega^2 \right)}_{\lambda_n} a_n^2 \end{aligned}$$

c) The integral measure can be expressed as (accept the following as a postulate, but think about it!)

$$Dy = J \prod_{n=1}^{\infty} da_n \quad (12)$$

for some constant J . Knowing this, show that

$$F_0(T) = \int Dy e^{iSy} = J \prod_{n=1}^{\infty} \sqrt{\frac{2\pi i}{m\lambda_n}} \quad (13)$$

The above equality can be computed using the Gaussian integral:

$$\begin{aligned} F_0(T) &= \int Dy e^{iSy} = J \int \prod_{n=1}^{\infty} da_n e^{i \frac{m}{2} \sum_{n=1}^{\infty} \lambda_n a_n^2} = J \int \prod_{n=1}^{\infty} da_n e^{i \frac{m}{2} \lambda_n a_n^2} \\ &= J \prod_{n=1}^{\infty} \int da_n e^{i \frac{m}{2} \lambda_n a_n^2} = J \prod_{n=1}^{\infty} \sqrt{\frac{2\pi i}{m\lambda_n}} = J \prod_{n=1}^{\infty} \sqrt{\frac{2\pi i}{m\lambda_n}} \end{aligned}$$

f) We know the exact value of $F_0(T)$ for the case of free fields, $\omega=0$:

Recall indeed that in this case $F_0(T) = \sqrt{\frac{m}{2\pi i T}}$. On the other hand, one

can also calculate $F_0(T)$ by the same procedure we developed until now:

Show that the λ_n coefficients, when $\omega=0$, read $\lambda_n^{(0)} = \frac{n^2 \pi^2}{T^2}$

We looked at free fields ($\omega=0$) just before part a) of this problem and found that

$$\begin{aligned} F_0(T) &= \int Dy e^{iSy} \stackrel{q_1(T)}{=} e^{-iS(q_1)} \int Dq e^{iS(q)} = e^{-iS(q_1)} \langle q_1, T | q_1, 0 \rangle \\ &= e^{-iS(q_1)} \langle q_1 | e^{-i\hat{H}T} | q_1 \rangle = e^{-iS(q_1)} \langle q_1 | e^{-i \frac{p^2}{2m} T} | q_1 \rangle \\ &= e^{-iS(q_1)} \sqrt{\frac{m}{2\pi i T}} e^{i \frac{m}{2T} (q_1 - q_1)^2} \stackrel{S(q_1)|_{\omega=0}, \text{ see eq. (8)}}{=} \sqrt{\frac{m}{2\pi i T}} \end{aligned}$$

Since $\lambda_n = \frac{n^2 \pi^2}{T^2} - \omega^2$, we immediately have $\lambda_n^{(0)} = \lambda_n|_{\omega=0} = \frac{n^2 \pi^2}{T^2}$.

g) Then we can write

$$\frac{F_0(T)}{F_0(T)} = \prod_{n=1}^{\infty} \sqrt{\frac{\lambda_n^{(0)}}{\lambda_n}} = \prod_{n=1}^{\infty} \left(1 - \frac{\omega^2 T^2}{n^2 \pi^2} \right)^{-\frac{1}{2}} \quad (14)$$

Deduce from this that

$$F_n(T) = \sqrt{\frac{m\omega}{2\pi i \sin(\omega T)}}$$

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and finally, collect everything and write up the result for the transition amplitude!

We first investigate the infinite product in eq. (14). Interestingly, it vanishes,

whenever $T = \frac{\pi n}{\omega}$ with $n \in \mathbb{N}$. The same applies to $\sin(\omega T)$ for $T = \frac{\pi n}{\omega}$.

However, for $T \rightarrow 0$, $\prod_{n=1}^{\infty} (1 - \frac{\omega^2 T^2}{\pi^2 n^2}) \rightarrow 1$ while $\sin(\omega T) \rightarrow 0$. So the infinite product might rather resemble $\frac{\sin(\omega T)}{\omega T}$ which we stated earlier behaves

as $\frac{\sin(\omega T)}{\omega T} \rightarrow 1$ for $T \rightarrow 0$. In fact, $\text{sinc}(\omega T) = \frac{\sin(\omega T)}{\omega T}$ is called the cardinal sine and the infinite product is exactly Euler's expansion of it.

The identity

$$\text{sinc}(\omega T) = \frac{\sin(\omega T)}{\omega T} = \prod_{n=1}^{\infty} (1 - \frac{\omega^2 T^2}{\pi^2 n^2})$$

can be proven rigorously (and kindly) using the residue theorem (or Liouville's theorem). We refrain from performing either proof and simply apply it to find

$$F_n(T) = F_n(T) \prod_{n=1}^{\infty} (1 - \frac{\omega^2 T^2}{\pi^2 n^2})^{-1/2} = F_n(T) \frac{1}{\sqrt{\text{sinc}(\omega T)}} = \sqrt{\frac{m}{2\pi i T}} \frac{1}{\sqrt{\text{sinc}(\omega T)}} = \sqrt{\frac{m\omega}{2\pi i \sin(\omega T)}}$$

Assembling everything we learned throughout this problem, we are finally able to write down the complete transition amplitude for the quantum harmonic oscillator from state q_n at $t=0$ to state q_n at $t=T$:

$$\begin{aligned} \langle q_n, T | q_n, 0 \rangle &= \int_{q_n}^{q_n} \mathcal{D}q e^{iS[q]} = e^{iS[q_n]} \int \mathcal{D}y e^{iS[y]} = e^{iS[q_n]} F_n(T) \\ &= \sqrt{\frac{m\omega}{2\pi i \sin(\omega T)}} e^{\frac{im\omega}{2\sin(\omega T)} ((q_n^2 + q_n^2) \cos(\omega T) - 2q_n q_n)} \end{aligned}$$