String Theory

Solution to Assignment 2

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1 The Polyakov action

The Polyakov action is given by

$$S_{\mathrm{P}} = -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}\tau\mathrm{d}\sigma\,\sqrt{-h}h^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}, \tag{1}$$

where h_{ab} is the world-sheet metric, h^{ab} is its inverse and $-h \equiv -\det(\boldsymbol{h})$. The quantity $G_{ab} \equiv \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$ is called the pull-back of the target space metric η_{ab} on to the world-sheet.

a) For an arbitrary square matrix $\boldsymbol{M}$, useful variation identities are

$$\begin{aligned}\delta(\det \boldsymbol{M}) &= \det(\boldsymbol{M})\,\mathrm{Tr}(\boldsymbol{M}^{-1}\delta\boldsymbol{M}),\\ &= -\det(\boldsymbol{M})\,\mathrm{Tr}(\boldsymbol{M}\,\delta\boldsymbol{M}^{-1}),\end{aligned} \tag{2}$$

$$\delta\boldsymbol{M}^{-1} = -\boldsymbol{M}^{-1}\delta\boldsymbol{M}\,\boldsymbol{M}^{-1}. \tag{3}$$

Show this, where for simplicity you may assume that $\boldsymbol{M}$ is diagonalizable.

b) Use part a) to show that the energy momentum tensor, defined by

$$T_{ab} \equiv \frac{4\pi}{\sqrt{-h}}\frac{\delta S_{\mathrm{P}}}{\delta h^{ab}}, \tag{4}$$

is given by

$$T_{ab} = -\frac{1}{\alpha'}\left(\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} - \frac{1}{2}h_{ab}h^{cd}\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu}\right). \tag{5}$$

c) The equation of motion for the world sheet metric h_{ab} implies $T_{ab} = 0$. Use this to show the equivalence of the Polyakov action and the Nambu-Goto action.

d) Neglecting boundary terms, show that the equation of motion for X^μ is given by

$$h^{ab}\nabla_a(\partial_b X^\mu) = 0. \tag{6}$$

Note: Take this opportunity to recall that for a vector S^a one has

$$\nabla_a(\sqrt{-\det(h)}Sa^) = \partial_a(\sqrt{-\det(h)}S^a), \tag{7}$$

and that $\nabla_a(\det h) = 0$.

e) Use part d) to show the conservation of energy and momentum,

$$\nabla^a T_{ab} = 0. \tag{8}$$

f) Use the conformal invariance of the action to show that the energy momentum tensor is traceless. Do not use the equation of motion for X^μ.

g) Now suppose we add to the Polyakov action (1) the 2-dimensional cosmological constant term

$$S_{\rm cc} = \lambda_{\rm cc} \int_\Sigma \mathrm{d}\tau \mathrm{d}\sigma \sqrt{-h}. \tag{9}$$

Show that consistency of the equations of motion for the metric h_{ab} requires $\lambda_{\rm cc} = 0$.

a) Note that requiring $\boldsymbol{M}$ to be square and diagonalizable is not sufficient for the identities (2) and (3) to be true in general, since $\boldsymbol{M}$ may have a zero eigenvalue on the diagonal, in which case $\det(\boldsymbol{M}) = 0$ and $\boldsymbol{M}$ is not invertible. A trivial example for this is the zero matrix $\mathbf{0}$ which is obviously square and diagonal but for which identities (2) and (3) are undefined. We proceed here by assuming that $\boldsymbol{M}^{-1}$ exists.

The determinant $\det(\boldsymbol{M})$ is a function of all elements of $\boldsymbol{M}$, i.e. if $\boldsymbol{M}$ is $n \times n$,

$$\det(\boldsymbol{M}) = \det(\boldsymbol{M})(M_{11}, M_{12}, \ldots, M_{nn}). \tag{10}$$

Therefore, by the chain rule, we have

$$\delta \det(\boldsymbol{M}) = \frac{\partial \det(\boldsymbol{M})}{\partial M_{11}} \delta M_{11} + \ldots \frac{\partial \det(\boldsymbol{M})}{\partial M_{nn}} \delta M_{nn} = \sum_{i,j=1}^{n} \frac{\partial \det(\boldsymbol{M})}{\partial M_{ij}} \delta M_{ij}. \tag{11}$$

The cofactor expansion (a.k.a. Laplace's formula) for the determinant can be taken by summing over the matrix entries M_{ij} and their corresponding cofactors C_{ij} of any *one* row i or column j, i.e.

$$\det(\boldsymbol{M}) = \sum_{j=1}^{n} M_{ij} C_{ij}, \tag{12}$$

where $C_{ij} = \mathrm{adj}(\boldsymbol{M})_{ji}$. Inserting this into $\frac{\partial \det(\boldsymbol{M})}{\partial M_{ij}}$ from the r.h.s of eq. (11) and executing the product rule, we get

$$\frac{\partial \det(\boldsymbol{M})}{\partial M_{ij}} = \frac{\partial}{\partial M_{ij}} \Big(\sum_{k=1}^{n} M_{ik} C_{ik} \Big) = \sum_{k=1}^{n} \frac{\partial M_{ik}}{\partial M_{ij}} C_{ik} + \sum_{k=1}^{n} M_{ik} \frac{\partial C_{ik}}{\partial M_{ij}}. \tag{13}$$

This expression drastically simplifies if we take into account that the cofactor C_{ik} of M_{ik} depends only on elements of $\boldsymbol{M}$ not in the same row or column as M_{ik}, i.e. $\frac{\partial C_{ik}}{\partial M_{ij}} = 0 \ \forall j, k$. Therefore

$$\frac{\partial \det(\boldsymbol{M})}{\partial M_{ij}} = \sum_{k=1}^{n} \underbrace{\frac{\partial M_{ik}}{\partial M_{ij}}}_{\delta_{j,k}} C_{ik} = C_{ij} = \mathrm{adj}(\boldsymbol{M})_{ji}. \tag{14}$$

Reinsertion of this result into eq. (11) yields

$$\delta \det(\boldsymbol{M}) = \sum_{i,j=1}^{n} \mathrm{adj}(\boldsymbol{M})_{ji} \, \delta M_{ij} = \mathrm{Tr}[\mathrm{adj}(\boldsymbol{M}) \, \delta \boldsymbol{M}]. \tag{15}$$

Since $\boldsymbol{M}$ is assumed to be invertible, we may write the adjoint as $\mathrm{adj}(\boldsymbol{M}) = \det(\boldsymbol{M})\boldsymbol{M}^{-1}$ and use linearity of the trace to arrive at

$$\delta \det(\boldsymbol{M}) = \mathrm{Tr}[\det(\boldsymbol{M})\, \boldsymbol{M}^{-1}\, \delta\boldsymbol{M}] = \det(\boldsymbol{M})\, \mathrm{Tr}[\boldsymbol{M}\, \delta\boldsymbol{M}^{-1}], \tag{16}$$

where in the last step, we used eq. (3) and cyclicity of the trace. Equation (3) follows directly from the product rule:

$$0 = \delta(\boldsymbol{M}^{-1}\, \boldsymbol{M}) = \delta\boldsymbol{M}^{-1}\, \boldsymbol{M} + \boldsymbol{M}^{-1}\, \delta\boldsymbol{M} \quad \Rightarrow \quad \delta\boldsymbol{M}^{-1} = -\boldsymbol{M}^{-1}\, \delta\boldsymbol{M}\, \boldsymbol{M}^{-1}. \tag{17}$$

b) By direct calculation, we get

$$\begin{aligned}
T_{ab}(\tau,\sigma) &= \frac{4\pi}{\sqrt{-h}} \frac{\delta S_{\mathrm{P}}}{\delta h^{ab}(\tau,\sigma)} \\
&= \frac{4\pi}{\sqrt{-h}} \frac{\delta}{\delta h^{ab}} \left(-\frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}\tau' \mathrm{d}\sigma'\, \sqrt{-h}\, h^{cd}(\tau',\sigma') \partial_c X^\mu(\tau',\sigma') \partial_d X^\nu(\tau',\sigma') \eta_{\mu\nu} \right) \\
&= -\frac{1}{\alpha'\sqrt{-h}} \int_\Sigma \mathrm{d}\tau' \mathrm{d}\sigma'\, \frac{\delta}{\delta h^{ab}} \left(\sqrt{-h}\, h^{cd} \right) \partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} \\
&= -\frac{1}{\alpha'\sqrt{-h}} \int_\Sigma \mathrm{d}\tau' \mathrm{d}\sigma' \left(\sqrt{-h}\, \frac{\delta h^{cd}(\tau',\sigma')}{\delta h^{ab}(\tau,\sigma)} - \frac{1}{2} \frac{1}{\sqrt{-h}} \frac{\delta h}{\delta h^{ab}}\, h^{cd} \right) \partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} \\
&\overset{\text{a)}}{=} -\frac{1}{\alpha'\sqrt{-h}} \int_\Sigma \mathrm{d}\tau' \mathrm{d}\sigma' \Bigg[\sqrt{-h}\, \delta_a{}^c \delta_b{}^d \delta(\tau-\tau')\delta(\sigma-\sigma') \\
&\qquad\qquad - \frac{1}{2} \frac{h^{cd}}{\sqrt{-h}} \Bigg(-h\, h_{ef} \underbrace{\frac{\delta h^{ef}(\tau',\sigma')}{\delta h^{ab}(\tau,\sigma)}}_{\delta_a{}^e \delta_b{}^f \delta(\tau-\tau')\delta(\sigma-\sigma')} \Bigg) \Bigg] \partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} \\
&= -\frac{1}{\alpha'\sqrt{-h}} \left[\sqrt{-h}\, \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} - \frac{1}{2} \frac{h^{cd}}{\sqrt{-h}} \left(-h\, h_{ab} \right) \partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} \right] \\
&= -\frac{1}{\alpha'} \left[\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} - \frac{1}{2} h_{ab} h^{cd} \partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} \right],
\end{aligned} \tag{18}$$

where we used identity (2) from part a) in the form of

$$\delta h = \delta(\det \boldsymbol{h}) = -\det(\boldsymbol{h})\, \mathrm{Tr}(\boldsymbol{h}\, \delta\boldsymbol{h}^{-1}) = -h\, h_{ef}\, \delta h^{ef}, \tag{19}$$

and refrained from writing out τ and σ dependencies in many places in order to save space.

c) The Nambu-Goto action is given by

$$S_{\mathrm{NG}} = -T \int_\Sigma \mathrm{d}\tau \mathrm{d}\sigma\, \sqrt{-\det(\boldsymbol{G})}, \tag{20}$$

where the $T = \frac{1}{2\pi\alpha'}$ is the string tension, and the components of the 2×2-matrix $\boldsymbol{G}$ can be calculated via

$$G_{ab} = \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}, \qquad a, b \in \{\tau, \sigma\}. \tag{21}$$

$\boldsymbol{G}$ is called the induced metric or the pullback of the ambient space metric $\eta_{\mu\nu}$ onto the string worldsheet Σ, i.e. it describes the embedding of a string in spacetime. Expressed in terms of T and $\boldsymbol{G}$, the Polyakov action reads

$$S_{\mathrm{P}} = -\frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}\tau \mathrm{d}\sigma\, \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} = -\frac{T}{2} \int_\Sigma \mathrm{d}\tau \mathrm{d}\sigma\, \sqrt{-h}\, \underbrace{h^{ab} G_{ab}}_{\boldsymbol{h}\boldsymbol{G}}. \tag{22}$$

By its very definition as given in eq. (4), the energy-momentum tensor T_{ab} is required to vanish by the equation of motion for h_{ab} resulting from an application of Hamilton's principle. Going back to our result from part b), we see that $T_{ab} = 0$ together with eq. (18) implies

$$\begin{aligned} T_{ab} = -\frac{1}{\alpha'}\Big(\underbrace{\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}}_{G_{ab}} - \frac{1}{2} h_{ab} h^{cd} \underbrace{\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu}}_{G_{cd}}\Big) = 0 \\ \Rightarrow \quad G_{ab} = \frac{1}{2} h_{ab}\, h^{cd} G_{cd} = \frac{1}{2} h_{ab}\, \boldsymbol{h}\,\boldsymbol{G}. \end{aligned} \tag{23}$$

Taking the determinant of eq. (23), we get

$$\det(\boldsymbol{G}) = \big(\tfrac{1}{2}\boldsymbol{h}\,\boldsymbol{G}\big)^2 \det(\boldsymbol{h}), \tag{24}$$

where the square around $\frac{1}{2}\boldsymbol{h}\,\boldsymbol{G}$ appears because the worldsheet is two-dimensional. Inserting this result into the Nambu-Goto action (20), we indeed arrive at the Polyakov action:

$$S_{\text{NG}} = -T \int_\Sigma \mathrm{d}\tau \mathrm{d}\sigma \sqrt{-\big(\tfrac{1}{2}\boldsymbol{h}\,\boldsymbol{G}\big)^2 \det(\boldsymbol{h})} = -\frac{T}{2}\int_\Sigma \mathrm{d}\tau\mathrm{d}\sigma\, \sqrt{-h}\, \boldsymbol{h}\,\boldsymbol{G} = S_{\text{P}}. \tag{25}$$

d) We calculate the string field's equation of motion by variation of the action:

$$\begin{aligned} \frac{\delta S_{\text{P}}}{\delta X_\mu} &= \frac{\delta}{\delta X_\mu}\left(-\frac{T}{2}\int_\Sigma \mathrm{d}\tau'\mathrm{d}\sigma'\, \sqrt{-h}\, h^{ab}\partial_a X_\nu \partial_b X^\nu\right) \\ &= -T\int_\Sigma \mathrm{d}\tau'\mathrm{d}\sigma'\, \sqrt{-h}\, h^{ab}\partial_b X^\nu \frac{\delta(\partial_a X_\nu)}{\delta X_\mu} \\ &= T\int_\Sigma \mathrm{d}\tau'\mathrm{d}\sigma'\, \partial_a\Big(\sqrt{-h}\, h^{ab}\partial_b X^\nu\Big) \underbrace{\frac{\delta X_\nu}{\delta X_\mu}}_{\delta^\mu{}_\nu \delta(\tau-\tau')\delta(\sigma-\sigma')} - \underbrace{T\sqrt{-h}\, h^{ab}\partial_a X^\nu \frac{\delta X_\nu}{\delta X_\mu}\bigg|_{\partial\Sigma}}_{\text{boundary term assumed to vanish}} \\ &= T\,\partial_a\Big(\sqrt{-h}\, h^{ab}\partial_b X^\mu\Big) \overset{!}{=} 0. \end{aligned} \tag{26}$$

From exercise 2.a) on assignment 1, we know that for any tensor S_{ab}, the determinant's square root $\sqrt{\det(\boldsymbol{S})}$ is a *scalar* density of weight $w = 1$. Therefore, the quantity $\sqrt{-h}\, h^{ab}\partial_b X^\mu$ is a *tensor* density of weight 1. We would like to compute its covariant derivative.

The covariant derivative $\nabla_a T^{b_1\ldots b_r}{}_{c_1\ldots c_s}$ of an arbitrary rank-(r,s) tensor $T^{b_1\ldots b_r}{}_{c_1\ldots c_s}$ is given by its partial derivative $\partial_a T^{b_1\ldots b_r}{}_{c_1\ldots c_s}$ plus an upstairs contraction with the connection for every contravariant index, $\Gamma^{b_1}{}_{ad} T^{d\ldots b_r}{}_{c_1\ldots c_s}$, and minus a downstairs contraction for every covariant index, $\Gamma^d{}_{ac_1} T^{b_1\ldots b_r}{}_{d\ldots c_s}$. The covariant derivative of a *tensor density* of weight w is defined in exactly the same way, except that we also add a term scaled with w where the connection is contracted with itself, i.e. $w\Gamma^d{}_{ad} T^{b_1\ldots b_r}{}_{c_1\ldots c_s}$.

Applying this prescription to $\sqrt{-h}\, h^{ab}\partial_b X^\mu$, we get

$$\nabla_a(\sqrt{-h}\, h^{ab}\partial_b X^\mu) = \partial_a(\sqrt{-h}\, h^{ab}\partial_b X^\mu) + \Gamma^a{}_{ac}\sqrt{-h}\, h^{cb}\partial_b X^\mu - 1\cdot\Gamma^c{}_{ca}\sqrt{-h}\, h^{ab}\partial_b X^\mu. \tag{27}$$

All the indices appearing in eq. (27) are summed over, so we may rename as we like. Exchanging a and c in the last term, we saw that it exactly cancels the second. We are left with $\nabla_a(\sqrt{-h}\, h^{ab}\partial_b X^\mu) = \partial_a(\sqrt{-h}\, h^{ab}\partial_b X^\mu)$. Using this result on our e.o.m. (26), we obtain

$$0 = \nabla_a(\sqrt{-h}\, h^{ab}\partial_b X^\mu) = \sqrt{-h}\, h^{ab}\nabla_a(\partial_b X^\mu), \tag{28}$$

since the metric connection satisfies $\nabla_a h_{bc} = 0$ (by the chain rule, the same holds for $\sqrt{-h}$ and any other function of h_{ab}). Dividing by the uninteresting prefactor $\sqrt{-h}$ which does not affect dynamics, we arrive at

$$h^{ab}\nabla_a(\partial_b X^\mu) = 0. \tag{29}$$

e) Conservation of energy and momentum again follows from direct computation,

$$
\begin{aligned}
\nabla^a T_{ab} &= -\frac{1}{\alpha'}\nabla^a\left(\partial_a X^\mu \partial_b X^\mu - \frac{1}{2}h_{ab}\partial_c X_\mu \partial^c X^\mu\right)\\
&= -\frac{1}{\alpha'}\Big[\underbrace{(\nabla^a\partial_a X_\mu)}_{=0,\text{ by part d)}}\partial_b X^\mu + \partial_a X_\mu(\nabla^a\partial_b X^\mu)\\
&\qquad - \frac{1}{2}h_{ab}\Big(\underbrace{(\nabla^a\partial_c X_\mu)\partial^c X^\mu + \partial_c X_\mu(\nabla^a\partial^c X^\mu)}_{2\partial_c X_\mu(\nabla^a\partial^c X^\mu)}\Big)\Big]\\
&= -\frac{1}{\alpha'}\Big[\partial_a X_\mu(\nabla^a\partial_b X^\mu) - \underbrace{\partial_c X_\mu(\nabla_b\partial^c X^\mu)}_{c\to a}\Big]\\
&= -\frac{1}{\alpha'}\partial_a X_\mu\Big[\nabla^a\partial_b X^\mu - \nabla_b\partial^a X^\mu\Big] = 0,
\end{aligned}
\tag{30}
$$

where in the last step, the two terms in brackets cancel because

$$
\begin{aligned}
\nabla^a\partial_b X^\mu &= h^{ac}\nabla_c\partial_b X^\mu = h^{ac}(\partial_c\partial_b X^\mu - \Gamma^d{}_{cb}\partial_d x^\mu)\\
&= h^{ac}(\partial_b\partial_c X^\mu - \Gamma^d{}_{bc}\partial_d x^\mu) = h^{ac}\nabla_b\partial_c X^\mu = \nabla_b\partial^a X^\mu.
\end{aligned}
\tag{31}
$$

f) Before proceeding with the exercise, it is important in this context to understand the difference between a conformal and a Weyl transformation, especially since in the lecture these two completely different concepts have been treated as one and the same.

Note:

1. A conformal transformation is a spacetime transformation which leaves the metric invariant up to (generally spacetime-dependent) scaling. The important property here is that angles are preserved.
2. A Weyl transformation actively scales the metric.

Formally, the difference can be expressed by considering two manifolds M and N with inner products, i.e. metrics, $\boldsymbol{g}$ and $\boldsymbol{h}$, and coordinates x^i and y^i, respectively. A map $f : M \to N$ from one manifold to another is called conformal if there exists a function $\Omega \in C^\infty(M)$ so that the pullback Ωg fulfills

$$f * \boldsymbol{h} = \Omega \boldsymbol{g}, \tag{32}$$

which in coordinate notation reads

$$\frac{\partial y^i}{\partial x^r}\frac{\partial y^j}{\partial x^s}h_{ij}(y) = \Omega(x)\, g_{rs}(x). \tag{33}$$

In case of conformal transformations, we have $M = N$ and thus equal metrics $\boldsymbol{g} = \boldsymbol{h}$ so that eq. (32) becomes

$$f * \boldsymbol{g} = \Omega \boldsymbol{g}, \tag{34}$$

or written in terms of the coordinates,

$$\frac{\partial y^i}{\partial x^r}\frac{\partial y^j}{\partial x^s}g_{ij}(y) = \Omega(x) g_{rs}(x), \tag{35}$$

which is just a coordinate transformation $x \to y$.
In case of Weyl transformations, we are simply scaling the metric, so here we also have $M = N$. However, since we are not changing coordinates this time around, the map f will simply be given by $f = \mathrm{id}_M$, yielding

$$h(x) = \Omega(x) g(x), \tag{36}$$

or in coordinates,

$$h_{ij}(x) = \Omega(x) g_{ij}(x). \tag{37}$$

After this short interlude, we note that even though the exercise mentions conformal invariance, we think it actually means what we have just distinguished to be Weyl invariance. From section 2.2.3 of the lecture notes, We know that the Polyakov action (1) features a local Weyl invariance. The metric on the other hand explicitly changes under a Weyl transformation[1]. The same holds for its inverse and its determinant. They transform as

$$h_{ab} \to e^{2\Lambda(\tau,\sigma)} h_{ab}, \qquad h^{ab} \to e^{-2\Lambda(\tau,\sigma)} h^{ab}, \qquad \det(\boldsymbol{h}) \to e^{2d\Lambda(\tau,\sigma)} \det(\boldsymbol{h}), \tag{38}$$

respectively, where d is the dimensionality of the space in which the determinant is taken, i.e. $d = 2$ on the worldsheet.

Going back to eq. (4), we see that under Weyl transformations, the energy-momentum tensor remains invariant,

$$T_{ab} \to T'_{ab} = \frac{4\pi}{\sqrt{-h'}} \frac{\delta S'_{\mathrm{P}}}{\delta h'^{ab}} = \frac{4\pi}{\sqrt{-e^{4\Lambda} h}} \frac{\delta S_{\mathrm{P}}}{e^{-2\Lambda} \delta h^{ab}} = \frac{4\pi}{\sqrt{-h}} \frac{\delta S_{\mathrm{P}}}{\delta h^{ab}} = T_{ab}. \tag{39}$$

Next, we check the trace $T^a{}_a$,

$$T^a{}_a = h^{ab} T_{ab} \to T'^a{}_a = h'^{ab} T'_{ab} = e^{-2\Lambda} h^{ab} T_{ab} = e^{-2\Lambda}\, T^a{}_a. \tag{40}$$

A priori, it seems like $T^a{}_a$ may be variant under Weyl rescalings. But if T_{ab} as a whole remains invariant, the same must apply to the trace $T^a{}_a$. This presents a contradiction unless the energy-momentum tensor is in fact traceless, $T^a{}_a = 0$.

g) Adding the cosmological constant term (9) to the Polyakov action (1), we get

$$S_{\mathrm{P+cc}} = S_{\mathrm{P}} + S_{\mathrm{cc}} = -\frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}\tau \mathrm{d}\sigma\, \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} + \lambda_{\mathrm{cc}} \int_\Sigma \mathrm{d}\tau \mathrm{d}\sigma\, \sqrt{-h}. \tag{41}$$

The metric's equation of motion for this action is

$$\begin{aligned}
\frac{\delta S_{\mathrm{P+cc}}}{\delta h^{ab}} &= \frac{\delta S_{\mathrm{P}}}{\delta h^{ab}} + \frac{\delta S_{\mathrm{cc}}}{\delta h^{ab}} = \frac{\sqrt{-h}}{4\pi} T_{ab} + \lambda_{\mathrm{cc}} \int_\Sigma \mathrm{d}\tau' \mathrm{d}\sigma' \left(-\frac{1}{2} \frac{1}{\sqrt{-h}} \frac{\delta h}{\delta h^{ab}} \right) \\
&\overset{(2)}{=} \frac{\sqrt{-h}}{4\pi} T_{ab} - \frac{\lambda_{\mathrm{cc}}}{2} \int_\Sigma \mathrm{d}\tau' \mathrm{d}\sigma' \left(\frac{-h}{\sqrt{-h}}\, h_{ef} \underbrace{\frac{\delta h^{ef}(\tau',\sigma')}{\delta h^{ab}(\tau,\sigma)}}_{\delta_a{}^e \delta_b{}^f \delta(\tau-\tau')\delta(\sigma-\sigma')} \right) \\
&= \frac{\sqrt{-h}}{4\pi} T_{ab} - \frac{\lambda_{\mathrm{cc}}}{2} \sqrt{-h}\, h_{ab} \overset{!}{=} 0,
\end{aligned} \tag{42}$$

where we again used the variational identity (2) to calculate the term $\frac{\delta h}{\delta h^{ab}}$ and obtain the second line. In the above, T_{ab} is the energy-momentum tensor of the unmodified Polyakov action S_{P}. Equation (42) implies

$$T_{ab} = 2\pi \lambda_{\mathrm{cc}} h_{ab}. \tag{43}$$

Taking the trace of this expression and using our result from part f), we get

$$0 = T^a{}_a = 2\pi\lambda_{\mathrm{cc}} \underbrace{h^a{}_a}_{2} = 4\pi\lambda_{\mathrm{cc}} \quad \Rightarrow \quad \lambda_{\mathrm{cc}} \overset{!}{=} 0. \tag{44}$$

2 Gravity in two dimensions is trivial

a) Using the symmetries of its indices, convince yourself that in two dimensions, the Riemann tensor has only one independent degree of freedom.

[1] In fact, the change in the metric is the root cause for all changes that may occur under a Weyl transformation in other quantities if they depend on the metric

b) Verify that the ansatz

$$R_{abcd} = \lambda(h_{ac}h_{bd} - h_{ad}h_{bc}) \tag{45}$$

with h_{ab} the two-dimensional metric is consistent with the symmetries of the Riemann tensor. Show that $\lambda = \frac{1}{2}\mathcal{R}$ in terms of the Ricci scalar $\mathcal{R}$.

c) Compute the Einstein tensor defined in terms of the Ricci tensor R_{ab} as

$$G_{ab} = R_{ab} - \frac{1}{2}h_{ab}\mathcal{R} \tag{46}$$

for a two-dimensional metric. Discuss the result.

d) Show that in two dimensions under the Weyl rescaling $h_{ab} \to e^{2\omega(\tau,\sigma)}h_{ab}$ the product $\sqrt{-\det(h)}\mathcal{R}$ transforms as

$$\sqrt{-\det(h)}\mathcal{R} \to \sqrt{-\det(h')}\mathcal{R}' = \sqrt{-\det(h)}[\mathcal{R} - 2\nabla^2\omega]. \tag{47}$$

e) Argue from the above result that the 2-dimensional Einstein-Hilbert term is indeed conformally invariant for a closed string worldsheet.

Note: By contrast, for an open string worldsheet Σ with boundary $\partial\Sigma$ only the combination

$$\chi = \frac{1}{4\pi}\int_\Sigma \mathrm{d}^2\xi\sqrt{-h}\mathcal{R} + \frac{1}{2\pi}\int \mathrm{d}s\mathcal{K} \tag{48}$$

is conformally invariant. Here the extrinsic curvature $\mathcal{K}$ is defined as

$$\mathcal{K} = \pm t^a n_b \nabla_a t^b, \tag{49}$$

with t^a a unit vector tangent to the boundary and n^a an outward unit vector orthogonal to t^a. The upper/lower sign refer to timelike/spacelike boundaries. Indeed this object is the **Euler characteristic of a worldsheet with boundary**.

f) Use the result from part d) to show that *locally* every metric of signature $(-1,1)$ can be brought into the form $\mathrm{diag}(\eta) = (-1,1)$ by Weyl rescalings and diffeomorphism invariance.

a) The components of the Riemann tensor can be conveniently calculated with the formula

$$R^a{}_{bcd} = \partial_c\Gamma^a{}_{bd} - \partial_d\Gamma^a{}_{bc} + \Gamma^e{}_{bd}\Gamma^a{}_{ce} - \Gamma^e{}_{bc}\Gamma^a{}_{de}. \tag{50}$$

As discussed in exercise 1 on assignment 1, in d dimensions the Riemann tensor has $n = d^4$ components which makes for a total of $n = 16$ if $d = 2$. To see how many of those components are actually independent degrees of freedom we need to consider the Riemann tensor's plentiful symmetries.

> **Note:** Not all of them are present in the above mixture of co- and contravariant indices, however. For a tensor, say the energy momentum tensor T_{ab}, to be e.g. symmetric under the exchange of two indices, $T_{ab} = T_{ba}$, both indices obviously need to be of the same variance. $T_a{}^b = T_b{}^a$ is strictly impossible since a tensor covariant in a can never be equal to one contravariant in a. They have completely different transformation properties.
> So in order to make all symmetries manifest, we lower the first index by contraction with the metric h_{ab}:
>
> $$R_{abcd} = h_{ae}R^e{}_{bcd} = h_{ae}\Big(\partial_c\Gamma^e{}_{bd} - \partial_d\Gamma^e{}_{bc} + \Gamma^f{}_{bd}\Gamma^e{}_{cf} - \Gamma^f{}_{bc}\Gamma^e{}_{df}\Big). \tag{51}$$
>
> The Riemann tensor is a field, i.e. it depends on the point in space we are looking at. To

simplify the derivation of its symmetry properties, we go about inspecting it from a locally inertial frame (LIF). An LIF has the special property that at its origin all metric derivatives vanish. It is common known ledge that, for a smooth metric, we can define a coordinate system at any given point so that locally the metric expressed in this coordinate system is the flat spacetime metric η_{ab}. What is less well known is that this operation does not exhaust the transformational degrees of freedom of the metric. It can be shown with some work that the remaining freedom suffices to impose all derivatives of the metric to vanish at a particular point in space.

From the definition

$$\Gamma^{a}{}_{bc} = \frac{1}{2} h^{ad} \left(\partial_c h_{bd} + \partial_b h_{cd} - \partial_d h_{bc} \right), \tag{52}$$

it is clear that vanishing metric derivatives result in all of the Christoffel symbols being zero as well. However, *second* derivatives of the metric need not vanish at the origin of an LIF. Hence, derivatives of the Christoffel symbols are also unequal to zero in general. Bearing that in mind, the Riemann tensor at the origin of an LIF takes the form

$$R_{abcd} = h_{ae} \Big(\partial_c \Gamma^{e}{}_{bd} - \partial_d \Gamma^{e}{}_{bc} \Big). \tag{53}$$

Explicit calculation of the derivative of the Christoffel symbol yields

$$\partial_a \Gamma^{b}{}_{cd} = \frac{1}{2} \underbrace{\partial_a h^{be}}_{0} \Big(\partial_d h_{ce} + \partial_c h_{de} - \partial_e h_{cd} \Big) + \frac{1}{2} h^{be} \Big(\partial_a \partial_d h_{ce} + \partial_a \partial_c h_{de} - \partial_a \partial_e h_{cd} \Big). \tag{54}$$

By contraction with the metric h_{bf} and using $h_{bf} h^{be} = \delta^{e}{}_{f}$, we get

$$\begin{aligned} h_{fb} \partial_a \Gamma^{b}{}_{cd} &= \frac{1}{2} h_{bf} h^{be} \Big(\partial_a \partial_d h_{ce} + \partial_a \partial_c h_{de} - \partial_a \partial_e h_{cd} \Big) \\ &= \frac{1}{2} \Big(\partial_a \partial_d h_{cf} + \partial_a \partial_c h_{df} - \partial_a \partial_f h_{cd} \Big). \end{aligned} \tag{55}$$

Adjusting index naming, we can insert this expression for both terms on the r.h.s. of eq. (53) to obtain

$$R_{abcd} = \frac{1}{2} \Big[\partial_c \partial_d h_{ba} + \partial_c \partial_b h_{da} - \partial_c \partial_a h_{bd} - \Big(\partial_d \partial_c h_{ba} + \partial_d \partial_b h_{ca} - \partial_d \partial_a h_{bc} \Big) \Big]. \tag{56}$$

Since partial derivatives commute, the first and fourth term cancel, and we are left with

$$\begin{aligned} R_{abcd} &= \frac{1}{2} \Big[\partial_c \partial_b h_{da} - \partial_c \partial_a h_{bd} - \partial_d \partial_b h_{ca} + \partial_d \partial_a h_{bc} \Big] \\ &= \frac{1}{2} \Big[\partial_a \partial_d h_{bc} - \partial_b \partial_d h_{ac} + \partial_b \partial_c h_{ad} - \partial_a \partial_c h_{bd} \Big], \end{aligned} \tag{57}$$

where in the second step all we did was some housekeeping. Remember that this equation is valid only at the origin of an LIF. You might then rightfully ask what good it is. The answer to that is simple: The origin of a LIF defines one particular point in spacetime. Since all symmetries were expressed i.t.o. tensor equations, they must be true at that point, regardless of which coordinate system we're using. Further, one can show that we could define an LIF with its origin at any point in spacetime, provided that point is locally flat (i.e. there is no singularity at that point). So the symmetries are in fact true for all non-singular points in spacetime[2].

[2]It might be confusing that we say that the symmetries in eq. (57) are universally valid at all points in all coordinate systems just because they are tensor equations, while at the same time maintain that eq. (57) is valid only at the origin of an LIF. The difference is that eq. (57) is written explicitly in terms of a particular metric h_{ab} which is defined precisely so that all its first derivatives are zero at the origin of the LIF. If we wanted an equation for R_{abcd} at some other point in spacetime, we could write it in the same form, but we'd need to find a different metric $h_{ab}{}'$ whose first

Taking a closer look at the index structure in eq. (57), we see that the Riemann tensor is antisymmetric both under exchange of the first and last two indices, i.e.

$$R_{abcd} = -R_{bacd}, \qquad R_{abcd} = -R_{abdc}. \tag{58}$$

That means of the original $d^2 = 4$ degrees of freedom that the first and last two indices contain separately, antisymmetry leaves us with only $\frac{d}{2}(d-1) = 1$ independent degree of freedom. Consequently, the Riemann tensor as a whole also only has $1 \cdot 1 = 1$ degree of freedom if $d = 2$.

b) The Riemann tensor, irrespective of the dimensionality of space, has four symmetries, two of which we just exploited, i.e. antisymmetry under exchange of the first and last two indices. The other two are symmetry under exchange of both pairs as a whole and the Bianchi identity. We will check each of these in turn for the ansatz (45):

1. $R_{abcd} = -R_{bacd}$?

$$R_{abcd} = \lambda(h_{ac}h_{bd} - h_{ad}h_{bc}) = -\lambda(h_{bc}h_{ad} - h_{bd}h_{ac}) = -R_{bacd}. \quad \checkmark \tag{59}$$

2. $R_{abcd} = -R_{abdc}$?

$$R_{abcd} = \lambda(h_{ac}h_{bd} - h_{ad}h_{bc}) = -\lambda(h_{ad}h_{bc} - h_{ac}h_{bd}) = -R_{abdc}. \quad \checkmark \tag{60}$$

3. $R_{abcd} = R_{cdab}$? This follows directly from symmetry of the metric:

$$R_{abcd} = \lambda(h_{ac}h_{bd} - h_{ad}h_{bc}) = \lambda(h_{ca}h_{db} - h_{da}h_{cd}) = R_{cdab}. \quad \checkmark \tag{61}$$

4. Bianchi identity: $R_{a[bcd]} = 0$?

$$\begin{aligned} R_{a[bcd]} &= R_{abcd} + R_{acdb} + R_{adbc} \\ &= \lambda(h_{ac}h_{bd} - \underline{h_{ad}h_{bc}} + \underline{\underline{h_{ab}h_{dc}}} - h_{ac}h_{db} + \underline{h_{ad}h_{cb}} - \underline{\underline{h_{ab}h_{cd}}}) = 0. \quad \checkmark \end{aligned} \tag{62}$$

Contraction of all indices yields

$$\mathcal{R} = R^{ab}{}_{ab} = \lambda(\underbrace{h^a{}_a}_{2}\underbrace{h^b{}_b}_{2} - \underbrace{h^a{}_b h^b{}_a}_{h^a{}_a}) = \lambda(4-2) = 2\lambda \quad \Rightarrow \quad \lambda = \frac{1}{2}\mathcal{R}. \tag{63}$$

c) The components of the Ricci tensor can be computed from those of the Riemann tensor by contracting the first and third index,

$$R_{ab} = R^c{}_{acb} = \lambda(h^c{}_c h_{ab} - h^c{}_b h_{ac}) = \lambda(2h_{ab} - h_{ab}) = \lambda h_{ab}. \tag{64}$$

Insertion into eq. (46) gives a vanishing Einstein tensor,

$$G_{ab} = R_{ab} - \frac{1}{2}h_{ab}\mathcal{R} = \lambda h_{ab} - \frac{1}{2}h_{ab}\, 2\lambda = 0. \tag{65}$$

This is the reason why gravity in two dimension is said to be trivial: The Einstein field equations $G_{ab} = 8\pi G\, T_{ab}$ reduce to $T_{ab} = 0$.

d) Deriving the transformation law (47) is tedious work. One has to go through the usual procedure in general relativity to investigate the affect from a change to the metric on the curvature:

1. Identify the new metric.
2. Calculate the corresponding Christoffel symbols.

derivatives are zero at this other point. If we wanted to use the original metric, then since this other point is not at the origin of the original LIF, the Christoffel symbols would not be zero at this point, and the expression for R_{abcd} would be much more complicated.

3. Derive the new Riemann tensor.
4. Finally, contract fully to get the transformed Ricci scalar.

Step 1 is a given as the new metric ${h_{ab}}'(\tau,\sigma) = e^{2\omega(\tau,\sigma)} h_{ab}(\tau,\sigma)$ was already written in the exercise. Step 2 requires actual work:

$$\begin{aligned}
\Gamma'^a{}_{bc} &= \frac{1}{2} h'^{ad}\Big(\partial_c {h_{bd}}' + \partial_b {h_{cd}}' - \partial_d {h_{bc}}'\Big) = \frac{1}{2} e^{-2\omega} h^{ad}\Big(\partial_c(e^{2\omega} h_{bd}) + \partial_b(e^{2\omega} h_{cd}) - \partial_d(e^{2\omega} h_{bc})\Big) \\
&= \frac{1}{2} e^{-2\omega} h^{ad} e^{2\omega}\Big(\partial_c h_{bd} + \partial_b h_{cd} - \partial_d h_{bc} + h_{bd}\partial_c 2\omega + h_{cd}\partial_b 2\omega - h_{bc}\partial_d 2\omega\Big) \\
&= \frac{1}{2} h^{ad}\big(\partial_c h_{bd} + \partial_b h_{cd} - \partial_d h_{bc}\big) + h^{ad}\big(h_{bd}\partial_c\omega + h_{cd}\partial_b\omega - h_{bc}\partial_d\omega\big) \\
&= \Gamma^a{}_{bc} + \big(\delta_b{}^a\partial_c + \delta_c{}^a\partial_b - h_{bc}\partial^a\big)\omega \equiv \Gamma^a{}_{bc} + C^a{}_{bc},
\end{aligned} \tag{66}$$

where for the purpose of preserving readability in the sequel, we defined the object

$$C^a{}_{bc} = \big(\delta_b{}^a\partial_c + \delta_c{}^a\partial_b - h_{bc}\partial^a\big)\omega. \tag{67}$$

Step 3, obtaining the new Riemann tensor, requires the bulk of the calculations,

$$\begin{aligned}
R'^a{}_{bcd} &= \partial_c\Gamma'^a{}_{bd} - \partial_d\Gamma'^a{}_{bc} + \Gamma'^e{}_{bd}\Gamma'^a{}_{ce} - \Gamma'^e{}_{bc}\Gamma'^a{}_{de} \\
&= \partial_c(\Gamma^a{}_{bd} + C^a{}_{bd}) - \partial_d(\Gamma^a{}_{bc} + C^a{}_{bc}) \\
&\quad + (\Gamma^e{}_{bd} + C^e{}_{bd})(\Gamma^a{}_{ce} + C^a{}_{ce}) - (\Gamma^e{}_{bc} + C^e{}_{bc})(\Gamma^a{}_{de} + C^a{}_{de}) \\
&= R^a{}_{bcd} + \partial_c C^a{}_{bd} - \partial_d C^a{}_{bc} \\
&\quad + \Gamma^e{}_{bd}C^a{}_{ce} + \Gamma^a{}_{ce}C^e{}_{bd} + C^e{}_{bd}C^a{}_{ce} - \Gamma^e{}_{bc}C^a{}_{de} - \Gamma^a{}_{de}C^e{}_{bc} - C^e{}_{bc}C^a{}_{de} \\
&\equiv R^a{}_{bcd} + U^a{}_{bcd},
\end{aligned} \tag{68}$$

or rather it would, if we didn't cheat by again defining a new object $U^a{}_{bcd}$ to contain the entire change of the Riemann tensor under Weyl transformations. With eq. (68), we can move on to step 4 and derive the Weyl transformed Ricci scalar by contracting all indices, i.e.

$$\mathcal{R}' = R'^{ab}{}_{ab} = h'^{bc}R'^a{}_{cab} = e^{-2\omega}h^{bc}(R^a{}_{cab} + U^a{}_{cab}) = e^{-2\omega}(\mathcal{R} + h^{bc}U^a{}_{cab}). \tag{69}$$

Equation (69) shows that the Ricci scalar is not only rescaled but also receives an offset under Weyl transformations. All that remains is to calculate this offset which is where the little cheat we took above will cost us since we need to simplify

$$\begin{aligned}
h^{bc}U^a{}_{cab} = h^{bc}\Big(&\partial_a C^a{}_{cb} - \partial_b C^a{}_{ca} + \Gamma^e{}_{cb}C^a{}_{ae} + \Gamma^a{}_{ae}C^e{}_{cb} \\
&+ C^e{}_{cb}C^a{}_{ae} - \Gamma^e{}_{ca}C^a{}_{be} - \Gamma^a{}_{be}C^e{}_{ca} - C^e{}_{ca}C^a{}_{be}\Big).
\end{aligned} \tag{70}$$

Again in the interest of avoiding long blocks of calculation, we handle each of the above terms separately. Our work will be greatly accelerated by noting that $C^a{}_{bc}$ is symmetric under exchange of the two lower indices[3] and contraction of the upper index with either yields

$$C^a{}_{ab} = C^a{}_{ba} = \big(\delta_a{}^a\partial_b + \delta_b{}^a\partial_a - h_{ab}\partial^a\big)\omega = \big(2\partial_b + \partial_b - \partial_b\big)\omega = 2\partial_b\omega. \tag{71}$$

Another useful identity is $C^a{}_{bc}$ contracted with the metric h^{bc} as this simply gives zero,

$$h^{bc}C^a{}_{bc} = h^{bc}\big(\delta_b{}^a\partial_c + \delta_c{}^a\partial_b - h_{bc}\partial^a\big)\omega = \big(\partial^a + \partial^a - 2\partial^a\big)\omega = 0. \tag{72}$$

[3] As it has to be in order for the Weyl transformed Christoffel symbols to uphold this symmetry.

With these shortcuts, we proceed:

$$h^{bc}\partial_a C^a{}_{cb} = h^{bc}\partial_a\big(\delta_b{}^a\partial_c + \delta_c{}^a\partial_b - h_{bc}\partial^a\big)\omega = \big(\partial_b\partial^b + \partial_c\partial^c - 2\partial_a\partial^a\big)\omega = 0, \tag{73}$$

$$h^{bc}\partial_b C^a{}_{ca} \overset{(71)}{=} \partial^c(2\partial_c\omega) = 2\partial^2\omega, \tag{74}$$

$$h^{bc}\Gamma^e{}_{cb}C^a{}_{ae} \overset{(71)}{=} 2h^{bc}\Gamma^e{}_{cb}\partial_e\omega, \tag{75}$$

$$h^{bc}\Gamma^a{}_{ae}C^e{}_{cb} \overset{(72)}{=} 0, \tag{76}$$

$$h^{bc}C^e{}_{cb}C^a{}_{ae} \overset{(72)}{=} 0, \tag{77}$$

$$\begin{aligned} h^{bc}\Gamma^e{}_{ca}C^a{}_{be} &= h^{bc}\Gamma^e{}_{ca}\big(\delta_b{}^a\partial_e + \delta_e{}^a\partial_b - h_{be}\partial^a\big)\omega = \big(h^{ac}\Gamma^e{}_{ca}\partial_e + h^{bc}\Gamma^a{}_{ca}\partial_b - \delta_e{}^c\Gamma^e{}_{ca}\partial^a\big)\omega \\ &= \big(h^{ac}\Gamma^e{}_{ca}\partial_e + \underbrace{\Gamma^a{}_{ca}\partial^c - \Gamma^c{}_{ca}\partial^a}_{0}\big)\omega = h^{ac}\Gamma^e{}_{ca}\partial_e\omega, \end{aligned} \tag{78}$$

$$h^{bc}\Gamma^a{}_{be}C^e{}_{ca} \overset{(78)}{=} h^{ac}\Gamma^e{}_{ca}\partial_e\omega, \tag{79}$$

$$\begin{aligned} h^{bc}C^e{}_{ca}C^a{}_{be} &= h^{bc}\big(\delta_c{}^e\partial_a + \delta_a{}^e\partial_c - h_{ca}\partial^e\big)\omega\,\big(\delta_b{}^a\partial_e + \delta_e{}^a\partial_b - h_{be}\partial^a\big)\omega \\ &= h^{bc}\big(\partial_b\omega\partial_c\omega + \partial_b\omega\partial_c\omega - h_{bc}\partial_a\omega\partial^a\omega + \partial_b\omega\partial_c\omega + 2\partial_b\omega\partial_c\omega - \partial_c\omega\partial_b\omega \\ &\qquad - h_{cb}\partial^e\omega\partial_e - \partial_c\omega\partial_b\omega + \partial_b\omega\partial_c\omega\big) \\ &= 4\partial_b\omega\partial^b\omega - 2\partial_a\omega\partial^a\omega - 2\partial^e\omega\partial_e\omega = 0. \end{aligned} \tag{80}$$

Reinserting eqs. (73) to (80) back into eq. (70), we get an offset of

$$\begin{aligned} h^{bc}U^a{}_{cab} &= -2\partial^2\omega + 2h^{bc}\Gamma^e{}_{cb}\partial_e\omega - h^{ac}\Gamma^e{}_{ca}\partial_e\omega - h^{ac}\Gamma^e{}_{ca}\partial_e\omega \\ &= -2h^{bc}(\partial_b\partial_c\omega - \Gamma^e{}_{cb}\partial_e\omega) = -2h^{bc}\nabla_b\partial_c\omega = -2h^{bc}\nabla_b\nabla_c\omega = -2\nabla^2\omega, \end{aligned} \tag{81}$$

where in the last step $\partial_c\omega = \nabla_c\omega$ because ω is just a scalar function for which partial and covariant derivative are identical.

Plugging our hard-earned result into eq. (69), we get exactly the transformation behavior as given in the exercise, i.e.

$$\begin{aligned} \sqrt{-\det(h)}\mathcal{R} \to \sqrt{-\det(h')}\mathcal{R}' &= e^{2\omega}\sqrt{-\det(h)}e^{-2\omega}[\mathcal{R} - 2\nabla^2\omega] \\ &= \sqrt{-\det(h)}[\mathcal{R} - 2\nabla^2\omega]. \end{aligned} \tag{82}$$

e) The Einstein-Hilbert term reads

$$S_{\mathrm{EH}}[\boldsymbol{h}] = \frac{\lambda_{\mathrm{EH}}}{4\pi}\int_\Sigma \mathrm{d}\tau\mathrm{d}\sigma\sqrt{-\det(\boldsymbol{h})}\mathcal{R}. \tag{83}$$

Hence under Weyl transformations, the action changes as

$$S_{\mathrm{EH}}[\boldsymbol{h}] \to S_{\mathrm{EH}}[\boldsymbol{h}]' = \frac{\lambda_{\mathrm{EH}}}{4\pi}\int_\Sigma \mathrm{d}\tau\mathrm{d}\sigma\sqrt{-\det(\boldsymbol{h})}[\mathcal{R} - 2\nabla^2\omega] = S_{\mathrm{EH}}[\boldsymbol{h}] + \Delta S_{\mathrm{EH}}[\boldsymbol{h}]. \tag{84}$$

Since the Ricci scalar changes only by a total derivative, it is easy to show that $\Delta S_{\mathrm{EH}}[\boldsymbol{h}]$ vanishes for a closed string:

$$\begin{aligned} \Delta S_{\mathrm{EH}}[\boldsymbol{h}] &= -\frac{\lambda_{\mathrm{EH}}}{2\pi}\int_\Sigma \mathrm{d}\tau\mathrm{d}\sigma\sqrt{-\det(\boldsymbol{h})}\nabla^2\omega = -\frac{\lambda_{\mathrm{EH}}}{2\pi}\int_\Sigma \mathrm{d}*_2\,\mathrm{d}\omega \\ &= -\frac{\lambda_{\mathrm{EH}}}{2\pi}\int_{\partial\Sigma} *_2\mathrm{d}\omega = 0, \end{aligned} \tag{85}$$

where $*_2$ is the Hodge-Stern operator known from differential geometry, and we used that the worldsheet of a closed string has no boundary, i.e. $\partial\Sigma = 0$.

> **Note:** For an open string, $\Delta S_{\mathrm{EH}}[\boldsymbol{h}]$ also vanishes but only after incorporating the extrinsic curvature term as mentioned around eq. (48) into the full action.

f) By solving

$$\mathcal{R}'(\tau,\sigma) = \mathcal{R}(\tau,\sigma) - 2\nabla^2\omega(\tau,\sigma) = 0, \tag{86}$$

for $\omega(\tau,\sigma)$, we can find a special $\omega_0(\tau,\sigma)$ for every point (τ,σ) on the worldsheet, so that locally, $\mathcal{R}(\tau,\sigma)$ and $-2\nabla^2\omega(\tau,\sigma)$ exactly cancel and we have $\mathcal{R}'(\tau,\sigma) = 0$.

In part a), we demonstrated that the Riemann tensor in two dimensions only has one degree of freedom. In part b), we showed that this degree of freedom is directly proportional to the Ricci scalar. Bearing that in mind, it is clear that $\mathcal{R}'(\tau,\sigma) = 0$ implies

$$R'^{a}{}_{bcd}(\tau,\sigma) = 0 \qquad \text{at the point } (\tau,\sigma)\ \forall\, a,b,c,d \in \{\tau,\sigma\}. \tag{87}$$

In words: For a two-dimensional space, we can always use a spacetime dependent rescaling $\omega(\tau,\sigma)$ of distances to locally remove any curvature and obtain flat space.

Using the remaining diffeomorphism invariance, i.e. invariance of the action under coordinate transformations, we can reshape the metric. Note that diffeomorphisms do not enable us to change the metric's signature. The eigenvalues of a matrix are a fundamental property that are shared among all representations of a matrix connected via coordinate transformations. However, if we already start from a Lorentzian metric $\boldsymbol{h}$ with signature $(-1,1)$, we can always find a coordinate transformation $\boldsymbol{P}$,

$$x^a \to x'^a = P^a{}_b\, x^b, \qquad x \in \{\tau,\sigma\}. \tag{88}$$

whose rows, or equivalently columns, are given by the eigenvectors of the metric $\boldsymbol{h}$. This matrix $\boldsymbol{P}$ then fulfills the famous identity

$$\boldsymbol{h} = \boldsymbol{P}\,\boldsymbol{h}_D\,\boldsymbol{P}^{-1}, \tag{89}$$

where $\boldsymbol{h}_D$ is the metric in diagonal shape, i.e.

$$\boldsymbol{h}_D \propto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{90}$$