

Group Theory - Problem Sheet 2Exercise 1 (Representations)

Recall the following notions from the lecture:

a) What is a representation?

A (linear) representation of a group G on a vector space V over a field K is a group homomorphism from G to $GL(V)$, i.e. a map

$$\rho: G \longrightarrow GL(V) = \text{Aut}(V) = \{\text{all invertible, linear transf. } V \rightarrow V\}$$

such that $\rho(g_1)\rho(g_2) = \rho(g_1 g_2) \quad \forall g_1, g_2 \in G$.

V is called the representation space and $\dim(V)$ the dimension of the representation.

b) What is a homomorphism (intertwiner) between two repr.?

A group homomorphism between two groups is a map that preserves the group operation, i.e. given two groups $(G, \cdot)$ and $(H, *)$, a group homomorphism φ with

$$\varphi: G \longrightarrow H$$

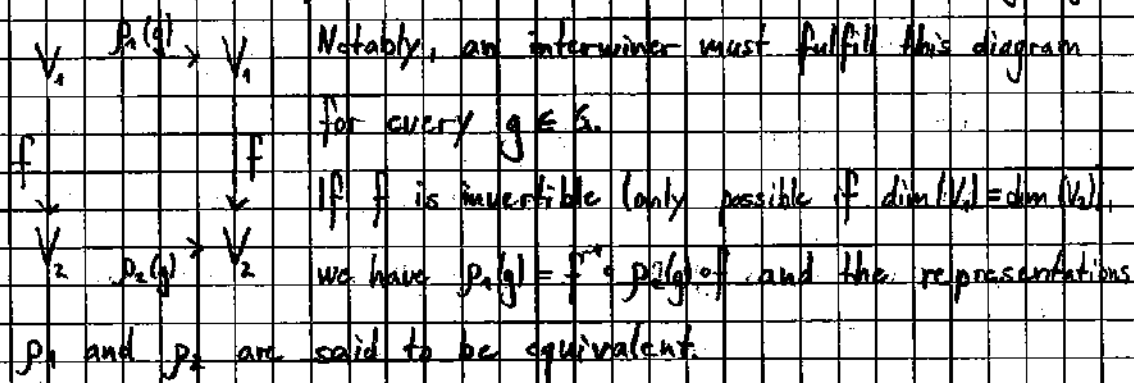
fulfills $\varphi(g_1 g_2) = \varphi(g_1) * \varphi(g_2) \quad \forall g_1, g_2 \in G$.

Let V_1 and V_2 be the representation spaces of two (linear) representations of the group G ,

$$\rho_1: G \longrightarrow GL(V_1), \quad \rho_2: G \longrightarrow GL(V_2).$$

Then, a linear map $f: V_1 \rightarrow V_2$ is called an intertwiner of the representations ρ_1 and ρ_2 if it commutes under the group action of G , i.e. if $f \circ \rho_1(g) = \rho_2(g) \circ f \quad \forall g \in G$.

An intertwiner's property can be visualized by the following diagram.



c) What is a subrepresentation?

A subspace $W \subset V$ of the representation space V of a representation ρ is called a subrepresentation, if it is invariant under the group action of the represented group G , where invariant means that $gW \subset W \forall g \in G$ and $W \subset V$.

The restriction of ρ to W , $\rho|_W$, is also referred to as subrepresentation, if $W \subset V$ is invariant subspace: $\rho|_W: G \rightarrow GL(W)$ is then a repr. on its own.

d) What are reducible and irreducible representations?

A representation $\rho: G \rightarrow GL(V)$ is said to be irreducible if it has only the trivial subrepresentations $\{0\}$ and V . If there is non-trivial subrepresentation, ρ is reducible.

e) Recall Schur's lemma.

If ρ_1 and ρ_2 are two finite-dimensional irreducible representations of a group G and $f: V_1 \rightarrow V_2$ is an intertwiner between their representation spaces, then either $f = 0$ or f is invertible, i.e. an isomorphism.

In the special case of $V_1 \cong V_2$, it further yields $f = \lambda \text{id}_V$ (with $\lambda \in \mathbb{C}$), if V_1 and V_2 are vector spaces over $\mathbb{C}$.

f) Argue that complex representations of finite groups are fully decomposable.

A representation $\rho: G \rightarrow GL(V)$ of a group G is called fully-decomposable, if it can be decomposed into a direct sum of irreducible representations, $\rho = \rho_1 \oplus \dots \oplus \rho_n$ where $\rho_i, i \in \{1, \dots, n\}$ each irreducible.

The above claim follows from this argument:

Let $\rho: G \rightarrow U(V)$ be a reducible complex (also called unitary) representation (if it were irreducible, it would be fully-decomposable by definition).

$\Rightarrow \exists W \subset V$ invariant subspace

We can define $W^\perp = \{v \in V \mid \langle v, w \rangle = 0 \ \forall w \in W\}$.

Then, since

W invariant subspace of unitary repr.

$$\forall g \in G, \forall w \in W, \forall v \in W^\perp: 0 = \langle v, w \rangle \stackrel{!}{=} \langle v, \rho(g)w \rangle \stackrel{!}{=} \langle \rho(g)v, w \rangle,$$

we know that $W^\perp$ is an invariant subspace as well.

We infer that $\rho = \rho|_W \oplus \rho|_{W^\perp}$.

$\uparrow$ still unitary, therefore we may split again.

The above scheme can be applied repeatedly until the subrepresentations become irreducible. Finally

$$\rho = \bigoplus_i \rho_i, \text{ with } \rho_i \text{ irreducible } \forall i$$

One may say that irreducible repr. are the building blocks of unitary repr.

Exercise 2 (Representations of abelian groups)

a) Show that all irreducible representations of a finite group G are one-dimensional if and only if G is abelian. (Hint: What is the sum of the squares of the dimensions of the irreducible repr.?)

To be shown: $\dim(\rho_i) = 1 \quad \forall \rho_i: G \rightarrow GL(V_i) \text{ irreducible}, i \in \{1, \dots, n\}$
 $\iff G \text{ abelian}$

" $\implies$ ": Assume all irreducible repr. have $d_i = \dim(\rho_i) = 1$. Then we have

$$|G| = \sum_{i=1}^n d_i^2 = \sum_{i=1}^n 1 = n, \dots \text{ with } n: \text{number of irred. repr.}$$

We also have that

$$\sum_{i=1}^n d_i = \text{'number of conjugacy classes of } G' = n$$

Therefore, 'number of conjugacy classes' $= n = |G|$, i.e. every element of G forms its own conjugacy class.

$$\iff hgh^{-1} = g \quad \forall g, h \in G \implies hg = gh \quad \forall g, h \in G \iff G \text{ is abelian}$$

" $\impliedby$ ": Assume G is abelian $\iff gh = hg \quad \forall g, h \in G$. Then, since every repr. is a homomorphism, we have

$$\begin{array}{c} \text{homomorphism } G \text{ abelian} \\ \downarrow \\ \rho(g)\rho(g_1) = \rho(g, g_1) = \rho(g_1, g) = \rho(g_1)\rho(g) \end{array}$$

Applying the special case of Schur's lemma, which states that if $f: V \rightarrow V$ commutes with $\rho(g) \quad \forall g \in G$, i.e.

$$f \circ \rho(g) = \rho(g) \circ f \quad \forall g \in G \implies f = \lambda \text{id}_V \text{ for some } \lambda \in \mathbb{C},$$

we obtain $\rho(g) = \lambda_g \text{id}_V = \begin{pmatrix} \lambda_g & & 0 \\ & \ddots & \\ 0 & & \lambda_g \end{pmatrix} = \bigoplus_i \rho_i$, with ρ_i irreducible repr. of $\dim(\rho_i) = 1$.

b) What are the irreducible representations of the cyclic group $\mathbb{Z}_n$?

Since $\mathbb{Z}_n$ is abelian, we know from part a) that every irreducible subrepr. of $\mathbb{Z}_n$ is one-dimensional and that there are exactly $|\mathbb{Z}_n| = n$ of them.

Let $\text{id}_\epsilon^{1/n}$ denote the n -th root of unity in $\mathbb{C}$, then the map

$$\rho: \mathbb{Z}_n \rightarrow \mathbb{C}^*, \quad \rho(g^i) = \text{id}_\epsilon^{i/n} \quad \forall g^i \in \mathbb{Z}_n$$

is a group homomorphism. Since $\text{id}_\epsilon^{1/n}, k \in \{1, \dots, n\}$ are the first through n -th roots of unity in $\mathbb{C}$, the n irreducible subrepresentations of $\mathbb{Z}_n$ with dimension one are

$$\rho_k: G \rightarrow \mathbb{C}^*, \quad \rho_k(g^i) = \text{id}_\epsilon^{ik/n} = e^{2\pi i \frac{ik}{n}}, \quad \text{with } k \in \{1, \dots, n\}$$

Exercise 3 (Orthogonality relation)

Let (ρ_1, V_1) and (ρ_2, V_2) be two irreducible (finite-dimensional complex) representations of a finite group G . Choose bases of V_1 and V_2 and let $(\rho_1(g))_{ij}$ and $(\rho_2(g))_{ab}$ be the repr. matrices with respect to these bases. Then use Schur's lemma to show:

a) If ρ_1 and ρ_2 are inequivalent, then

$$\sum_{g \in G} (\rho_1(g^{-1}))_{ij} (\rho_2(g))_{ab} = 0$$

for all i, j, a, b .

To show this, we consider the matrix

$$M_X = \sum_{g \in G} \underbrace{(\rho_1(g))^{-1}}_{\rho_1(g^{-1})} X \rho_2(g),$$

where X is an arbitrary matrix with d_1 rows and d_2 columns, d_1 and d_2 being the dimensionalities of the representations

ρ_1 and ρ_2 , s.t. M_X is a $d_1 \times d_2$ -matrix.

Now multiplying M_X from the left and right with arbitrary matrices in the representations ρ_1 and ρ_2 respectively,

$$\begin{aligned} \rho_1^{-1}(h) M_X \rho_2(h) &= \sum_{g \in G} \underbrace{\rho_1^{-1}(h)}_{\rho_1(h^{-1})} \underbrace{\rho_1^{-1}(g)}_{\rho_1(g^{-1})} X \rho_2(g) \rho_2(h) \quad \text{for some } h \in G \\ &= \sum_{g \in G} \rho_1(h^{-1}g^{-1}) X \rho_2(gh) \quad g \mapsto gh^{-1} \\ &= \sum_{g \in G} \rho_1(g^{-1}) X \rho_2(g) = M_X. \end{aligned}$$

We find that M_X is an intertwiner between the repr. ρ_1 and ρ_2 .

i.e. $\rho_1(g) M_X = M_X \rho_2(g)$.

Since ρ_1 and ρ_2 are taken to be inequivalent, i.e.

$$\nexists M_X : \rho_1(g) = M_X \rho_2(g) M_X^{-1}, \quad (M_X \text{ not invertible})$$

We know by Schur's lemma that $M_X = 0$.

Therefore, writing element-wise, we have

$$0 = (M_X)_{ib} = \sum_{g \in G} (\rho_1(g^{-1}))_{ik} X_{kj} (\rho_2(g))_{lb}.$$

Since the matrix X was arbitrary, we may choose any form.

We pick one X for every tuple of j, a given by

$$(X_{kj})_{ja} = \delta_{kj} \delta_{ja}$$

to conclude that

$$\sum_{g \in G} (\rho_1(g^{-1}))_{ik} \delta_{kj} \delta_{ja} (\rho_2(g))_{lb} = \sum_{g \in G} (\rho_1(g^{-1}))_{ij} (\rho_2(g))_{la} \stackrel{!}{=} 0$$

for all i, j, a, b .

b) If on the other hand $\rho_1 = \rho_2$, then

$$\frac{1}{|G|} \sum_{g \in G} (\rho(g^{-1}))_{ij} (\rho(g))_{ab} = \frac{1}{\dim(V)} \delta_{ib} \delta_{ja}.$$

In the case of $\rho_1 = \rho_2 = \rho$, ρ is still an irreducible representation of G and M_X with

$$M_X = \sum_{g \in G} (\rho(g))^{-1} X \rho(g)$$

is still an intertwiner, so that we may apply Schur's lemma for the special case of $V_1 = V_2 =: V$ to conclude

$$M_X = \lambda \text{id}_V \quad \text{for some } \lambda \in \mathbb{C} \text{ for every } g \in G$$

Taking the trace of M_X , we obtain

$$\begin{aligned} \text{Tr}(M_X) &= \text{Tr}\left(\sum_{g \in G} \rho(g^{-1}) X \rho(g)\right) \stackrel{\text{linearity of trace}}{=} \sum_{g \in G} \text{Tr}_V(\rho(g^{-1}) X \rho(g)) \\ &\stackrel{\text{cyclicity of trace}}{=} \sum_{g \in G} \text{Tr}_V(\underbrace{\rho(g) \rho(g^{-1})}_X X) = \sum_{g \in G} \text{Tr}_V(\underbrace{\text{id}_V}_X X) = |G| \text{Tr}_V(X) \end{aligned}$$

$$\text{Tr}(M_X) = \text{Tr}(\lambda \text{id}_V) = \lambda \text{Tr}(\text{id}_V) = \lambda \dim(V) \Rightarrow \lambda = \frac{|G|}{\dim(V)} \text{Tr}_V(X)$$

Choosing again $(X_{ja})_{ki} = \delta_{kj} \delta_{ai}$, we obtain

$$\begin{aligned} (M_X)_{ib} &= \sum_{g \in G} (\rho(g^{-1}))_{ik} \delta_{kj} \delta_{ai} (\rho(g))_{ib} = \sum_{g \in G} (\rho(g^{-1}))_{ij} (\rho(g))_{ab} \\ &= \lambda \delta_{ib} = \frac{|G|}{\dim(V)} \text{Tr}_V(X) \delta_{ib} = \frac{|G|}{\dim(V)} \underbrace{\text{Tr}_V(\delta_{kj} \delta_{ai})}_{\delta_{kj} \delta_{ai} = \delta_{ja}} = \frac{|G|}{\dim(V)} \delta_{ib} \delta_{ja} \end{aligned}$$

$$\Rightarrow \frac{1}{|G|} \sum_{g \in G} (\rho(g^{-1}))_{ij} (\rho(g))_{ab} = \frac{1}{\dim(V)} \delta_{ib} \delta_{ja}$$

c) Deduce that

$$\frac{1}{|G|} \sum_{g \in G} \chi_{\rho_1}(g^{-1}h) \chi_{\rho_2}(g) = \begin{cases} \frac{\chi_{\rho_1}(h)}{\dim(\rho_1)} & \text{if } \rho_1 = \rho_2 \\ 0 & \text{otherwise} \end{cases}$$

We express the characters $\chi_{\rho_1}(g^{-1}h) := \text{Tr}_{V_1}(\rho_1(g^{-1}h))$ and $\chi_{\rho_2}(g) := \text{Tr}_{V_2}(\rho_2(g))$ as

$$\chi_{\rho_1}(g^{-1}h) = \sum_{i,j} (\rho_1(g^{-1}))_{ji} (\rho_1(h))_{ij} \quad \text{and}$$

$$\chi_{\rho_2}(g) = \sum_k (\rho_2(g))_{kk}$$

to obtain

$$\frac{1}{|G|} \sum_{g \in G} \chi_{\rho_1}(g^{-1}h) \chi_{\rho_2}(g) = \frac{1}{|G|} \sum_{g \in G} \sum_{i,j,k} (\rho_1(g^{-1}))_{ji} (\rho_1(h))_{ij} (\rho_2(g))_{kk}$$

$$= \frac{1}{|G|} \sum_{i,j,k} (\rho_1(h))_{ij} (\rho_2(h))_{kk} (\rho_1(h))_{ji}$$

$$= \begin{cases} \frac{1}{\dim(V)} \sum_{i,j,k} \delta_{ik} \delta_{jk} (\rho(h))_{ji} = \frac{1}{\dim(V)} \sum_k (\rho(h))_{kk} = \frac{\chi_{\rho}(h)}{\dim(V)} & \text{if } \rho_1 = \rho_2 =: \rho \text{ using a)} \\ \frac{1}{|G|} \sum_{i,j,k} 0 \cdot (\rho(h))_{ji} = 0 & \text{otherwise using a)} \end{cases}$$

Exercise 4 (Symmetric and antisymmetric products)

Let (ρ, V) be a finite dimensional representation. Consider the tensor product representation $\rho \otimes \rho$ on $V \otimes V$. Define $\sigma: V \otimes V \rightarrow V \otimes V$ by $\sigma(v_1 \otimes v_2) = v_2 \otimes v_1 \quad \forall v_1, v_2 \in V$.

a) Show that the tensor product decomposes into the sum

$V \otimes V = S^2 V \oplus \Lambda^2 V$ of spaces of symmetric and anti-symmetric tensors $S^2 V = \{v \in V \mid \sigma(v) = v\}$ and $\Lambda^2 V = \{v \in V \mid \sigma(v) = -v\}$, and that these are invariant

subspaces in $V \otimes V$. (Hint: Consider the projectors $\frac{1}{2}(1 \pm \sigma): V \otimes V \rightarrow V \otimes V$.)

Following the hint, we examine $\pi_{\pm} := \frac{1}{2}(\text{id}_{V \otimes V} \pm \sigma): V \otimes V \rightarrow V \otimes V$

$$\pi_{\pm}^2 = \frac{1}{4}(\text{id}_{V \otimes V} \pm \sigma)^2 = \frac{1}{4}(\underbrace{\text{id}_{V \otimes V}^2}_{\text{id}_{V \otimes V}} \pm 2 \underbrace{\text{id}_{V \otimes V} \sigma}_{\sigma} \pm \underbrace{\sigma^2}_{\text{id}_{V \otimes V}}) = \frac{1}{4}(2\text{id}_{V \otimes V} \pm 2\sigma) = \pi_{\pm}$$

As hinted, we find $\pi_{\pm}$ to be projectors. In addition,

$$\pi_+ + \pi_- = \frac{1}{2}(\text{id}_{V \otimes V} + \sigma) + \frac{1}{2}(\text{id}_{V \otimes V} - \sigma) = \text{id}_{V \otimes V}$$

Therefore, the tensor product $V \otimes V$ can be decomposed as

$$V \otimes V = \text{Im}(\pi_+) \oplus \text{Im}(\pi_-)$$

It remains to be shown that $\text{Im}(\pi_+) = S^2V$ and $\text{Im}(\pi_-) = \Lambda^2V$.

$$S^2V \subset \text{Im}(\pi_+): \forall v \in S^2V: \sigma(v) = v$$

$$\Rightarrow \forall v \in S^2V: \pi_+(v) = \frac{1}{2}(\text{id}_{V \otimes V}(v) + \sigma(v)) = \frac{1}{2}(v + v) = v$$

$$\Rightarrow S^2V \subset \text{Im}(\pi_+), \text{ since } \pi_+ \text{ is projector}$$

$$\text{Im}(\pi_+) \subset S^2V: \forall v \in \text{Im}(\pi_+): \pi_+(v) = v, \text{ since } \pi_+ \text{ is projector}$$

$$\Rightarrow \pi_+(v) = \frac{1}{2}(\text{id}_{V \otimes V}(v) + \sigma(v)) = \frac{1}{2}(v + \sigma(v)) = v \Leftrightarrow \sigma(v) = v$$

$$\Rightarrow \text{Im}(\pi_+) \subset S^2V$$

$$\Rightarrow \text{Im}(\pi_+) = S^2V$$

This goes through analogously for $\text{Im}(\pi_-) = \Lambda^2V$.

$$\Rightarrow V \otimes V = S^2V \oplus \Lambda^2V$$

To further show that S^2V and Λ^2V are invariant subspaces in

$V \otimes V$, we remember a remark from the lecture, which states that image and kernel of every intertwiner form invariant subspaces. Since σ is an intertwiner because it fulfills $\rho \circ \rho(g) \cdot \sigma = \sigma \circ \rho \circ \rho(g)$, so is $\pi_{\pm}$ w.r.t. $\rho \otimes \rho$.

Therefore, S^2V and Λ^2V are invariant subspaces.

b) Show that the characters of the corresponding subrepresentations can be expressed as $\chi_{S_V}(g) = \frac{1}{2}[(\chi_V(g))^2 + \chi_V(g^2)]$, $\chi_{A_V}(g) = \frac{1}{2}[(\chi_V(g))^2 - \chi_V(g^2)]$. (A character is defined as the trace over a representation $\chi_V(g) = \text{tr}_V(\rho(g))$.)

A subrepresentation is formed by restricting a representation ρ to a subspace W of its representation space V , $\rho|_W$.

Since $\pi_{\pm}$ are projectors onto their respective subspaces, we may write the subrepresentations $\rho \otimes \rho|_{S_V}$ and $\rho \otimes \rho|_{A_V}$ as

$$\rho \otimes \rho|_{S_V} = \rho \otimes \rho \pi_+, \quad \rho \otimes \rho|_{A_V} = \rho \otimes \rho \pi_-$$

Making use of $\pi_{\pm}$ also being intertwiners, we may express the characters by

$$\chi_{S_V}(g) = \text{tr}_{V \otimes V}(\pi_+ \rho \otimes \rho(g)) = \frac{1}{2} [\text{tr}_{V \otimes V}(\text{id}_{V \otimes V} \rho \otimes \rho(g)) + \text{tr}_{V \otimes V}(\sigma \rho \otimes \rho(g))]$$

$$\chi_{A_V}(g) = \text{tr}_{V \otimes V}(\pi_- \rho \otimes \rho(g)) = \frac{1}{2} [\text{tr}_{V \otimes V}(\text{id}_{V \otimes V} \rho \otimes \rho(g)) - \text{tr}_{V \otimes V}(\sigma \rho \otimes \rho(g))]$$

We can simplify both summands

$$\text{tr}_{V \otimes V}(\text{id}_{V \otimes V} \rho \otimes \rho(g)) = \text{tr}_V(\rho(g)) \text{tr}_V(\rho(g)) = [\text{tr}_V(\rho(g))]^2 = \chi_V^2(g)$$

To calculate the second, we need to choose a basis, say e_i , for V . Then

$$\begin{aligned} \text{tr}_{V \otimes V}(\sigma \rho \otimes \rho(g)) &= \text{tr}_{V \otimes V}(\sigma \rho(g) \otimes \rho(g)(e_i \otimes e_j)) \\ &= \text{tr}_{V \otimes V}(\rho(g) \otimes \rho(g)(e_j \otimes e_i)) = \text{tr}_{V \otimes V}(\rho_{ji}(g) \rho_{ij}(g) e^b \otimes e^a) \\ &= \text{tr}_V(\rho_{ji}(g) \rho_{ij}(g)) = \text{tr}_V(\rho(g)^2) \end{aligned}$$

Reinserting the simplifications into $\chi_{\text{sym}}(g)$ and $\chi_{\text{asym}}(g)$, we arrive at the desired result:

$$\chi_{\text{sym}}(g) = \frac{1}{2} [\chi_v^2(g) + \chi_v(g^2)]$$

$$\chi_{\text{asym}}(g) = \frac{1}{2} [\chi_v^2(g) - \chi_v(g^2)]$$

Exercise 4 (Representations of D_5)

Consider the dihedral group

$$D_5 = \{e, R_1, R_2, R_3, R_4, S_1, S_2, S_3, S_4, S_5\}$$

of symmetries of the regular pentagon in the plane. Its character table is given by

D_5	$C_1 = \{e\}$	$C_2 = \{R_1, R_4\}$	$C_3 = \{R_2, R_3\}$	$C_4 = \{S_1, \dots, S_5\}$
χ_1	1	1	1	1
χ_2	1	1	1	-1
χ_3	2	α	β	0
χ_4	2	β	α	0

- a) Use orthogonality of characters to show that $\alpha = \frac{1}{2}(-1 + \sqrt{5})$ and $\beta = \frac{1}{2}(-1 - \sqrt{5})$.

The orthogonality relation of two characters is given by

$$\sum_a \chi_i(C_a) \chi_j(C_a) \frac{|C_a|}{|G|} = \delta_{ij}, \quad \text{where } \overline{\chi(g)} = \chi(g^{-1})$$

However, using in this example the orthogonality of rows would yield the equally correct but undesired solution

$$\alpha = -3 \pm \sqrt{3}, \quad \beta = -1 \pm \sqrt{3}.$$

So we apply the orthog. rel. for columns given by

$$\sum_i \chi_i(C_a) \chi_i(C_b) = \frac{|G|}{|C_a|} \delta_{ab}$$

Multiplying the first and second (or third), and the second and third columns, we obtain the constraints

$$1 \cdot 1 + 1 \cdot 1 + 2\alpha + 2\beta = 2 + 2\alpha + 2\beta = 0 \iff 1 + \alpha + \beta = 0,$$

$$1 \cdot 1 + 1 \cdot 1 + \alpha\beta + \beta\alpha = 2 + 2\alpha\beta = 0 \iff 1 + \alpha\beta = 0,$$

$$\Rightarrow \beta = -(1 + \alpha), \quad 1 + \alpha\beta = 1 - \alpha(1 + \alpha) = 1 - \alpha - \alpha^2 = 0$$

$$\iff \left(\alpha + \frac{1}{2}\right)^2 = \frac{5}{4} \Rightarrow \alpha = -\frac{1}{2} \pm \sqrt{\frac{5}{4}} = \frac{1}{2}(-1 \pm \sqrt{5})$$

$$\Rightarrow \beta = -\left(1 + \frac{1}{2}(-1 \pm \sqrt{5})\right) = \frac{1}{2}(-1 \mp \sqrt{5})$$

b) Consider the following representations $\rho_2 \otimes \rho_2$, $\Lambda^2 \rho_2$, $S^2 \rho_2$.

Determine their characters and decompose them into irreducible representations.

Characters are multiplicative with respect to tensor product, i.e.

$$\chi_{\rho_2 \otimes \rho_2}(g) = \chi_{\rho_2}(g) \cdot \chi_{\rho_2}(g)$$

Applying our result from exercise 4.b) i.e.

$$\chi_{S^2 \rho_2}(g) = \frac{1}{2}(\chi_{\rho_2}^2(g) + \chi_{\rho_2}(g^2)), \quad \chi_{\Lambda^2 \rho_2}(g) = \frac{1}{2}(\chi_{\rho_2}^2(g) - \chi_{\rho_2}(g^2)),$$

we may write a new character table for $\chi_{\rho_2 \otimes \rho_2}$, $\chi_{S^2 \rho_2}$, $\chi_{\Lambda^2 \rho_2}$

D_5	C_1	C_2	C_3	C_5
$\chi_{\rho_2 \otimes \rho_2}$	4	-1	-1	0
$\chi_{S^2 \rho_2}$	3	$-\alpha$	$-\beta$	1
$\chi_{\Lambda^2 \rho_2}$	1	1	1	-1

Apparently, $\chi_{\rho_2 \otimes \rho_2} = \chi_1 + \chi_3$. Therefore, $\rho_2 \otimes \rho_2 \cong \rho_1 \oplus \rho_3$. Also

$$\chi_{S^2 \rho_2} = \chi_1 + \chi_3 \Rightarrow S^2 \rho_2 \cong \rho_1 \oplus \rho_3$$

$$\chi_{\Lambda^2 \rho_2} = \chi_1 \Rightarrow \Lambda^2 \rho_2 \cong \rho_1$$